UNIT I

Riemann Steiltjes integral

LESSON-1

DEFINITION AND EXISTENCE OF THE INTEGRAL

1.0 Introduction

In UG , you have studied the definition of Riemann integral and the existence theorems on it. Riemann Stieltjes integral is the extension of Riemann integral. In particular , if we set $\alpha(x)=x$ in Riemann Stieltjes integral, we get the Riemann integral.

1.1 Aims and objectives

After studying this lesson you should be able to

- Find whether the Riemann Stieltjes integral of a function exists
- Find the Riemann Stieltjes integral.

1.2 Definition of Riemann –Stieltjes integral and existence theorems.

Definition

Let [a, b] be a given interval. A partition P of [a, b] is a finite set of points $x_0, x_1, x_2, \dots x_n$ such that

$$\mathbf{a = x_0 \le x_1 \le x_2 \le \dots\dots\dots\dots \le x_n = b}.$$

Definition

Let α be a monotonically increasing function on [a, b].
Corresponding to any partition P of [a, b],

$$\alpha_i = \alpha(x_i) - \alpha(x_{i-1}), \qquad I=1,2,\dots n.$$

Then $\alpha_i \ge 0$.
Let f be a bounded real valued function on [a, b].

Let $$\mathbf{U(P,f,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i}$$

$$\mathbf{L(P,f,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i}$$

where $M_i=\text{Sup}\{ f(x) \ / \ x \in [x_{i-1}, x_i]\}$
and $m_i= \inf\{ f(x) \ / \ x \in [x_{i-1}, x_i]\}$

We define the upper Riemann-Stieltjes integral of as

$$\overline{\int_a^b} \mathbf{f d\alpha = \inf U(P,f,\alpha)}$$

and the lower Riemann –Stieltjes integral of f as

$$\underline{\int_a^b} \mathbf{f d\alpha} = \mathbf{\sup L(P, f, \alpha)},$$

where the infinimum and supremum are taken over all partitions P of [a, b].

$$\text{If } \underline{\int_a^b} \mathbf{f d\alpha} = \overline{\int_a^b} \mathbf{f d\alpha},$$

their common value is denoted by $\int_a^b \mathbf{f d\alpha}$ or $\int_a^b \mathbf{f(x) d\alpha(x)}$.

This is called the Riemann-Stieltjes integral of **f** with respect to **α** on [a, b].

If $\int_a^b \mathbf{f d\alpha}$ exists ,then **f** is said to be integrable with respect to **α** on [a, b].

It is written as $\mathbf{f} \in \mathbf{R(\alpha)}$ on [a, b].

Definition

A partition P* is said to be a refinement of P , if $\mathbf{P^* \supseteq P}$.

Remark:

Given two partitions P_1 and P_2 of [a, b], their common refinement is $P^*=P_1 \cup P_2$.

Theorem 1.1

If P* is a refinement of P, then

$$\mathbf{U(P^*, f, \alpha) \leq U(P, f, \alpha)}$$

and $\mathbf{L(P^*, f, \alpha) \geq L(P, f, \alpha)}$.

Proof:

Assume that P* contains just one point more than P.

Let this be c and $x_{i-1}<c<x_i$.

Let M_i'= Sup{ f(x) / $x \in [x_{i-1}, c]$}

and M_i''= Sup{ f(x) / $x \in [c, x_i]$}.

Then $Mi' \leq M_i$ and $M_i'' \leq M_i$.

Consider

$$\mathbf{U(P^*, f, \alpha)} = \sum_{\substack{k=1 \\ k \neq i}}^{n} \mathbf{M_k} \Delta\alpha_k + M_i'[\alpha(\mathbf{c}) - \alpha(\mathbf{x_{i-1}})] + M_i''[\alpha(\mathbf{x_i}) - \alpha(\mathbf{c})]$$

$$\leq \sum_{\substack{k=1 \\ k \neq i}}^{n} \mathbf{M_k} \Delta\alpha_k + M_i[\alpha(\mathbf{c}) - \alpha(\mathbf{x_{i-1}})] + M_i[\alpha(\mathbf{x_i}) - \alpha(\mathbf{c})]$$

$$\leq \sum_{\substack{k=1 \\ k \neq i}}^{n} \mathbf{M_k} \Delta\alpha_k + M_i[\alpha(\mathbf{x_i}) - \alpha(\mathbf{x_{i-1}})]$$

$$\leq \mathbf{U(P, f, \alpha)}.$$

Similarly we can prove that

$$\mathbf{L(P^*, f, \alpha) \geq L(P, f, \alpha)}.$$

Hence the theorem.

Theorem 1.2

$$\underline{\int_a^b} \mathbf{f} d\alpha \leq \overline{\int_a^b} \mathbf{f} d\alpha .$$

Proof:

Let P_1 and P_2 be any partitions of [a, b].

Let $P^* = P_1 \cup P_2$.

Then P^* is the common refinement of P_1 as well as P_2.

Therefore by theorem 1.1,

$$\mathbf{U(P^*, f, \alpha)} \leq \mathbf{U(P_1, f, \alpha)} \qquad \text{…..(1)}$$

and $$\mathbf{L(P^*, f, \alpha)} \geq \mathbf{L(P_2, f, \alpha)} \qquad \text{…..(2)}$$

Also we know that

$$\mathbf{L(P^*, f, \alpha)} \leq \mathbf{U(P^*, f, \alpha)} \qquad \text{…..(3)}$$

From (1), (2) and (3), we get

$$\mathbf{L(P_2, f, \alpha)} \leq \mathbf{L(P^*, f, \alpha)} \leq \mathbf{U(P^*, f, \alpha)} \leq \mathbf{U(P_1, f, \alpha)}$$

Therefore for any two partitions P_1 and P_2 of [a, b], we have

$$\mathbf{L(P_2, f, \alpha)} \leq \mathbf{U(P_1, f, \alpha)} .$$

Keeping P_2 fixed and varying P_1 over all partitions of [a, b],

$$\mathbf{L(P_2, f, \alpha)} \leq \inf \mathbf{U(P, f, \alpha)}$$

Now this is true for all partitions P_2 of [a, b].

Therefore ,

$$\text{Sup}\, \mathbf{L(P, f, \alpha)} \leq \inf\ \mathbf{U(P, f, \alpha)} .$$

Therefore

$$\underline{\int_a^b} \mathbf{f} d\alpha \leq \overline{\int_a^b} \mathbf{f} d\alpha .$$

Hence the theorem.

Thorem1.3

$\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, b] if and only if there exists a partition P of [a, b] such that

$$\mathbf{U(P, f, \alpha)} - \mathbf{L(P, f, \alpha)} < \varepsilon .$$

Proof:

Let $\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, b] .

Then $$\underline{\int_a^b} \mathbf{f} d\alpha = \overline{\int_a^b} \mathbf{f} d\alpha , \qquad \text{…(1)}$$

where $$\overline{\int_a^b} \mathbf{f} d\alpha = \mathbf{inf\ U(P, f, \alpha)}$$

and $$\underline{\int_a^b} \mathbf{f} d\alpha = \mathbf{sup\, L(P, f, \alpha)} ,$$

Therefore , by definition of infinimum and supremum,

for given $\varepsilon > 0$, there exists a partition P_1 of [a, b] such that

$$\mathbf{U(P_1, f, \alpha)} < \overline{\int_a^b} \mathbf{f} d\alpha + \varepsilon / 2 \qquad \text{...(2)}$$

and a partition P_2 of [a, b] such that

$$\mathbf{L(P_2, f, \alpha)} > \underline{\int_a^b} \mathbf{f} d\alpha - \varepsilon / 2. \qquad . \qquad \text{...(3)}$$

Let $P = P_1 \cup P_2$.

Then by theorem 1.1,

$$\mathbf{U(P, f, \alpha)} \leq \mathbf{U(P_1, f, \alpha)} \qquad \text{...(4)}$$

and $\mathbf{L(P,f,\alpha)} \geq \mathbf{L(P_2,f,\alpha)}$ ...(5)

Therefore ,

From (1), (2), (3), (4) and (5), we get

$$\mathbf{U(P,f,\alpha)} \leq \mathbf{U(P_1,f,\alpha)}$$
$$< \overline{\int_a^b} \mathbf{f}d\alpha + \varepsilon/2$$
$$< \overline{\int_a^b} \mathbf{f}d\alpha + \varepsilon/2$$
$$< \mathbf{L(P_2,f,\alpha)} + \varepsilon/2 + \varepsilon/2$$
$$< \mathbf{L(P,f,\alpha)} + \varepsilon$$

Therfore there exists a partition P of [a, b] such that

$$\mathbf{U(P,f,\alpha)} - \mathbf{L(P,f,\alpha)} < \varepsilon.$$

Conversely ,

assume that there exists a partition P of [a, b] such that

$$\mathbf{U(P,f,\alpha)} - \mathbf{L(P,f,\alpha)} < \varepsilon. \quad \ldots(6)$$

For every partition P of [a, b] , we have

$$\mathbf{L(P,f,\alpha)} \leq \underline{\int_a^b} \mathbf{f}d\alpha \leq \overline{\int_a^b} \mathbf{f}d\alpha \leq \mathbf{U(P,f,\alpha)} \quad \ldots(7)$$

From (6) and (7), we get that

$$0 \leq \overline{\int_a^b} \mathbf{f}d\alpha - \underline{\int_a^b} \mathbf{f}d\alpha \leq \mathbf{U(P,f,\alpha)} - \mathbf{L(P,f,\alpha)} < \varepsilon.$$

This is true for every $\varepsilon > 0$.

Hence $\overline{\int_a^b} \mathbf{f}d\alpha - \underline{\int_a^b} \mathbf{f}d\alpha = 0.$

Therefore,

$$\overline{\int_a^b} \mathbf{f}d\alpha = \underline{\int_a^b} \mathbf{f}d\alpha.$$

Hence $\mathbf{f} \in \mathbf{R(\alpha)}$ on [a, b].

Hence the theorem.

Theorem1.4

If **f** is continuous on [a, b], then $\mathbf{f} \in \mathbf{R(\alpha)}$ on [a, b].

Proof:

Let $\varepsilon > 0$.

Choose $\eta > 0$ such that $[\alpha(b) - \alpha(a)]\ \eta < \varepsilon$.

Since **f** is continuous on [a, b] and [a, b] is compact, **f** is uniformly continuous on [a, b].

Therefore for this $\eta > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(t)| < \eta \text{ whenever } x, t \in [a, b] \text{ with } |x-t| < \delta. \quad \ldots.(1)$$

If P is any partition of [a, b] such that $\Delta x_i < \delta$,

then $M_i - m_i = \sup \{ |f(x)-f(t)| \,/\, x, t \in [x_{i-1}, x_i]\}$

$\leq \eta$, $\quad$ i=1,2…n.

Therefore $\mathbf{U(P,f,\alpha)} - \mathbf{L(P,f,\alpha)} = \sum_{i=1}^{n} \mathbf{M}_i \Delta\alpha_i - \sum_{i=1}^{n} \mathbf{m}_i \Delta\alpha_i$

$$= \sum_{i=1}^{n} (\mathbf{M}_i - \mathbf{m}_i)\Delta\alpha_i$$

$$\leq \eta \sum_{i=1}^{n} \Delta\alpha_i$$

$$\leq \eta[\alpha(b) - \alpha(a)]$$

$$< \varepsilon .$$

Therefore $\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, b].

Hence the theorem.

Theorem 1.5

If f is monotonic on [a, b], and if α is continuous on [a, b], then $\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, b] .

Proof:

Let α be increasing on [a, b].

Let $\varepsilon > 0$ be given.

Choose n large enough such that

$$[\alpha(b) - \alpha(a)]/n < \varepsilon/[f(b) - f(a)].$$

Choose a partition P such that $\Delta\alpha_i = [\alpha(b) - \alpha(a)]/n$.

Let f be increasing on [a, b].

Hence $f(x_{i-1}) \leq f(x) \leq f(x_i)$ whenever $x_{i-1} \leq x \leq x_i$.

Therefore , $M_i = f(x_i)$ and $m_i = f(x_{i-1})$, $i=1,2,\ldots n$.

Therefore $$\mathbf{U(P,f,\alpha) - L(P,f,\alpha)} = \sum_{i=1}^{n} \mathbf{M}_i\Delta\alpha_i - \sum_{i=1}^{n} \mathbf{m}_i\Delta\alpha_i$$

$$= \sum_{i=1}^{n} (\mathbf{M}_i - \mathbf{m}_i)\Delta\alpha_i$$

$$= \sum_{i=1}^{n} (\mathbf{f(x_i) - f(x_{i-1})})[(\alpha(\mathbf{b}) - \alpha(\mathbf{a}))/\mathbf{n}]$$

$$= [(\alpha(\mathbf{b}) - \alpha(\mathbf{a}))/\mathbf{n}][\mathbf{f(b)\text{-}f(a)}]$$

$$< \varepsilon$$

Therefore $\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, b]

Hence the theorem.

Check your progress

1.Suppose $P_1=\{0,1/4,1/2,3/4,1\}$ and $P_2=\{0,1/3,2/3,1)$

Is P_1 a refinement of P_2?

2.Can you find a refinement that is finer than both P_1 and P_2?

3.Can you prove that $\int_a^b d\alpha(x) = \alpha(b) - \alpha(a)$, directly from the definition?

4. If f(x)=0 for all irrational x, f(x)=1 for all rational x, is $f \in R$ on [a, b] for any a<b.

1.3Let us sum up

In this lesson we have seen the definition of Riemann stieltjes integral, lower Riemann Stieltjes integral and the upper Riemann Stieltjes integral and that the

- Lower integral is always less than or equal to upper integral

And the necessary conditions for the existence of the integral are

- **f** is continuous on [a, b]
- f is monotonic on [a, b], and if α is continuous on [a, b]

and the necessary and sufficient condition is that

- there exists a partition P of [a, b] such that $\mathbf{U(P,f,\alpha)-L(P,f,\alpha)} < \varepsilon$.

1.4 Lesson End Activities

1. Suppose $f \geq 0$, f is contionous on [a,b,] and $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0\ \forall\ x \in [a,b]$.
2. Suppose $\propto$ is an increasing function on [a,b] and $\propto$ is contionous at $x_0 \in [a,b]$ & $f : [a,b] \mid\!\rightarrow R$ is defined by $f(x_0) = 1$ & $f(x) = 0$ $fx = x_0$. Prove that $f \in R(\propto)$ and $\int f dx = 0$.
3. Give an Example of a function in which $\underline{\int} fdx < \overline{\int} fdx$

1.5 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

LESSON - 2

PROPERTIES OF INTEGRALS

2.0 Introduction

In this lesson ,we are going to see the properties of Riemann Stieltjes integral.

2.1Aims and objectives

After studying this lesson you should be able to

- Identify the properties of the integral and
- use them to find the Riemann stieltjes integral of functions

2.2Properties of integral.

Theorem2.1

If $f_1 \in R(\alpha)$ and $f_2 \in R(\alpha)$ on [a, b], then $f_1 + f_2 \in R(\alpha)$ and

$$\int_a^b (f_1 + f_2)d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

Proof:

Let $f=f_1+f_2$ and P be any partition of [a, b].

Consider

$$U(P,f,\alpha) = \sum_{k=1}^{n} M_k \Delta\alpha_k$$

where $M_k=\sup\{f(x)/ x\in [x_{k-1}, x_k]\}$

$= \sup\{f_1(x)+f_2(x) \ / x\in [x_{k-1}, x_k]\}$

$\leq \sup\{f_1(x)/ x\in [x_{k-1}, x_k]\}+ \sup\{f_2(x)/ x\in [x_{k-1}, x_k]\}$

$\leq M_k'+M_k''$

Therefore ,

$$\sum_{k=1}^{n} M_k \Delta\alpha_k \leq \sum_{k=1}^{n} M_k' \Delta\alpha_k + \sum_{k=1}^{n} M_k'' \Delta\alpha_k$$

Therefore,

$$U(P,f,\alpha) \leq U(P,f_1,\alpha) + U(P,f_2,\alpha) \qquad ...(1)$$

Similarly, $$L(P,f,\alpha) \geq L(P,f_1,\alpha) + L(P,f_2,\alpha) \qquad ...(2)$$

Since $f_1 \in R(\alpha)$ on [a, b] and $f_2 \in R(\alpha)$ on [a, b],

for given $\varepsilon>0$, there exists partitions P_1 and P_2 such that

$$U(P_1,f_1,\alpha) - L(P_1,f_1,\alpha) < \varepsilon/2 \qquad …(3)$$

and $$U(P_2,f_2,\alpha) - L(P_2,f_2,\alpha) < \varepsilon/2 \qquad …(4)$$

Let $P=P_1 \cup P_2$

Then we know that

$$U(P,f_1,\alpha) \leq U(P_1,f_1,\alpha)$$

and $\mathbf{U(P,f_2,\alpha)} \le \mathbf{U(P_2,f_s,\alpha)}$

and $\mathbf{L(P,f_1,\alpha)} \ge \mathbf{L(P_1,f_1,\alpha)}$

and $\mathbf{L(P,f_2,\alpha)} \ge \mathbf{L(P_2,f_2,\alpha)}$.

Therefore ,

$$\mathbf{U(P,f_1,\alpha)} - \mathbf{L(P,f_1,\alpha)} \le \mathbf{U(P_1,f_1,\alpha)} - \mathbf{L(P_1,f_1,\alpha)}$$
$$< \varepsilon/2 \qquad \ldots(5)$$

and
$$\mathbf{U(P,f_2,\alpha)} - \mathbf{L(P,f_2,\alpha)} \le \mathbf{U(P_2,f_2,\alpha)} - \mathbf{L(P_2,f_2,\alpha)}$$
$$< \varepsilon/2 \qquad \ldots(6)$$

From (1), (2), (5) and (6), we get

$$\begin{aligned}\mathbf{U(P,f,\alpha)} - \mathbf{L(P,f,\alpha)} &\le [\,\mathbf{U(P,f_1,\alpha)} + \mathbf{U(P,f_2,\alpha)}\,] - [\,\mathbf{L(P,f_1,\alpha)} + \mathbf{L(P,f_2,\alpha)}\,]\\ &\le [\,\mathbf{U(P,f_1,\alpha)} - \mathbf{L(P,f_1,\alpha)}\,] + [\,\mathbf{U(P_2,f_2,\alpha)} - \mathbf{L(P_2,f_2,\alpha)}\,]\\ &< \varepsilon/2 \ + \varepsilon/2\\ &< \varepsilon .\end{aligned}$$

Hence $\mathbf{f=f_1+f_2} \in \mathbf{R(\alpha)}$ on [a, b].

For the same Partition P of [a, b],

$$\begin{aligned}\mathbf{U(P,f_1,\alpha)} &< \mathbf{L(P,f_1,\alpha)} + \varepsilon/2\\ &< \sup\{\,\mathbf{L(P,f_1,\alpha)}\,\} + \varepsilon/2\\ &< \underline{\int_a^b \mathbf{f_1 d\alpha}} + \varepsilon/2\\ &< \int_a^b \mathbf{f_1 d\alpha} + \varepsilon/2\end{aligned}$$

[Since $\mathbf{f_1} \in \mathbf{R(\alpha)}$ on [a, b], $\int_a^b \mathbf{f_1 d\alpha} = \overline{\int_a^b} \mathbf{f_1 d\alpha} = \underline{\int_a^b \mathbf{f_1 d\alpha}}$.]

Similarly $\mathbf{U(P,f_2,\alpha)} < \int_a^b \mathbf{f_2 d\alpha} + \varepsilon/2$

Therefore,

$$\begin{aligned}\int_a^b \mathbf{f d\alpha} &= \inf\{\,\mathbf{U(P,f,\alpha)}\,\}\\ &< \mathbf{U(P,f,\alpha)}\\ &< \mathbf{U(P,f_1,\alpha)} + \mathbf{U(P,f_2,\alpha)}\\ &< \int_a^b \mathbf{f_1 d\alpha} + \varepsilon/2 + \int_a^b \mathbf{f_2 d\alpha} + \varepsilon/2\\ &< \int_a^b \mathbf{f_1 d\alpha} + \int_a^b \mathbf{f_2 d\alpha} + \varepsilon\end{aligned}$$

Hence
$$\int_a^b \mathbf{f d\alpha} \le \int_a^b \mathbf{f_1 d\alpha} + \int_a^b \mathbf{f_2 d\alpha} \qquad \ldots.(7)$$

Replacing f_1 by $-f_1$ and f_2 by $-f_2$, we get

$$-\int_a^b \mathbf{f d\alpha} \le -\int_a^b \mathbf{f_1 d\alpha} - \int_a^b \mathbf{f_2 d\alpha}$$

Multiplying both sides by (-1), we get

$$\int_a^b \mathbf{f d\alpha} \ \ge \int_a^b \mathbf{f_1 d\alpha} \int_a^b \mathbf{f_2 d\alpha} \qquad \ldots.(8)$$

From (7) and (8), we get that

$$\int_a^b f d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

Hence the theorem.

Theorem2.2

If $f \in R(\alpha)$ on [a, b] ,

then $c f \in R(\alpha)$ on [a, b], for any constant c and

$$\int_a^b cf d\alpha = c\int_a^b f d\alpha .$$

Proof:

If c = 0, then the result is true.

Assume that c > 0.

Since $f \in R(\alpha)$ on [a, b],

for given ε >0, there exists a partition P of [a, b] such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon /c.$$

Consider

$$U(cP,f,\alpha) - L(cP,f,\alpha) = \sum_{k=1}^{n} M_k{}'\Delta\alpha_k - \sum_{k=1}^{n} m_k{}'\Delta\alpha_k$$

where $M_k' = \sup\{(cf)(x) \,/\, x \in [x_{k-1}, x_k]\}$

$= c \sup\{ f(x) \,/\, x \in [x_{k-1}, x_k]\}$

$= cM_k$

Similarly $m_k' = cm_k$

Therefore ,

$$U(cP,f,\alpha) = c\sum_{k=1}^{n} M_k \Delta\alpha_k = c\, U(P,f,\alpha)$$

and $$L(cP,f,\alpha) = c\sum_{k=1}^{n} m_k \Delta\alpha_k = c\, L(P,f,\alpha)$$

Therefore $U(cP,f,\alpha) - L(cP,f,\alpha) = c[\,U(P,f,\alpha) - L(P,f,\alpha)\,] < \varepsilon$.

Hence $cf \in R(\alpha)$ on[a, b].

Therefore for the same P,

$$\begin{aligned} U(cP,f,\alpha) &< L(cP,f,\alpha) + \varepsilon \\ &\le \sup L(cP,f,\alpha) + \varepsilon \\ &\le c\sup L(P,f,\alpha) + \varepsilon \\ &\le c\int_a^b f d\alpha + \varepsilon \end{aligned}$$

Therefore,

$$\begin{aligned} \inf U(cP,f,\alpha) &\le U(cP,f,\alpha) \\ &\le c\int_a^b f d\alpha + \varepsilon \end{aligned}$$

Therefore,

$$\int_a^b cf d\alpha \le c\int_a^b f d\alpha \qquad \ldots(1)$$

Replacing f by –f, we get

$$-\int_a^b cf d\alpha \le c[-\int_a^b f d\alpha]$$

Multiplying both sides by (-1), we get

$$\int_a^b \mathbf{cfd}\alpha \ge \mathbf{c}\int_a^b \mathbf{fd}\alpha \qquad \text{….(2)}$$

From (1) and (2),

$$\int_a^b \mathbf{cfd}\alpha = \mathbf{c}\int_a^b \mathbf{fd}\alpha .$$

Hence the theorem.

Theorem 2.3

If $\mathbf{f}_1 \in \mathbf{R}(\alpha)$ on [a, b] , $\mathbf{f}_2 \in \mathbf{R}(\alpha)$ on [a, b] and

$\mathbf{f_1(x) \le f_2(x)}$ **on [a, b]**,

then $\int_a^b \mathbf{f}_1\mathbf{d}\alpha \le \int_a^b \mathbf{f}_2\mathbf{d}\alpha$

Proof:

Let P be any partition of [a, b].

Since $f_1(x) \le f_2(x)$,

$$\text{Sup}\{f_1(x) / x \in [x_{k-1,} x_k]\} \le \sup\{ f_2(x) / x \in [x_{k-1,} x_k]\}$$

Therefore ,

$$\mathbf{U(P,f_1,\alpha) \le U(P,f_2,\alpha)}$$

Therefore,

$$\inf \mathbf{U(P,f_1,\alpha)} \le \inf \mathbf{U(P,f_2,\alpha)}$$

Therefore,

$$\overline{\int_a^b} \mathbf{f}_1\mathbf{d}\alpha \le \overline{\int_a^b} \mathbf{f}_2\mathbf{d}\alpha$$

Therefore ,

$$\int_a^b \mathbf{f}_1\mathbf{d}\alpha \le \int_a^b \mathbf{f}_2\mathbf{d}\alpha$$

[since $\mathbf{f}_1 \in \mathbf{R}(\alpha)$ on [a, b] , $\mathbf{f}_2 \in \mathbf{R}(\alpha)$ on [a, b]

$$\int_a^b \mathbf{f}_1\mathbf{d}\alpha = \overline{\int_a^b} \mathbf{f}_1\mathbf{d}\alpha \quad \text{and} \quad \int_a^b \mathbf{f}_2\mathbf{d}\alpha = \overline{\int_a^b} \mathbf{f}_2\mathbf{d}\alpha \quad].$$

Hence the theorem.

Theorem2.4

If $\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, b] and if a<c<b, then $\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, c] and $\mathbf{f} \in \mathbf{R}(\alpha)$ on [c, b], and

$$\int_a^c \mathbf{fd}\alpha + \int_c^b \mathbf{fd}\alpha = \int_a^b \mathbf{fd}\alpha .$$

Proof:

Since $\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, b], for given ε >o, there exists a partition P of [a, b] such that

$$\mathbf{U(P,f,\alpha) - L(P,f,\alpha)} < \varepsilon .$$

Let $P_1 = P \cap [a, c]$ and $P_2 = P \cap [c, b]$

The P_1 is a partition of [a, c] and P_2 is a partition of [c, b].

Also on [a, c],

$$\mathbf{U(P_1,f,\alpha) - L(P_1,f,\alpha)} < \mathbf{U(P,f,\alpha) - L(P,f,\alpha)} < \varepsilon .$$

and on [c, b],

$$\mathbf{U(P_2,f,\alpha) - L(P_2,f,\alpha)} < \mathbf{U(P,f,\alpha) - L(P,f,\alpha)} < \varepsilon$$

Therefore,

$$\mathbf{f} \in \mathbf{R}(\alpha) \text{ on [a, c] and [c, b].}$$

For any partition P of [a, b],

Since $P = P_1 \cup P_2$,

$$\begin{aligned} \mathbf{U(P,f,\alpha)} &= \mathbf{U(P_1,f,\alpha)} + \mathbf{U(P_2,f,\alpha)} \\ &\geq \mathbf{\inf U(P_1,f,\alpha)} + \mathbf{\inf U(P_2,f,\alpha)} \\ &= \overline{\int_a^c} \mathbf{fd\alpha} + \overline{\int_c^b} \mathbf{fd\alpha} \end{aligned}$$

Therefore,

$$\inf \mathbf{U(P,f,\alpha)} \geq \overline{\int_a^c} \mathbf{fd\alpha} + \overline{\int_c^b} \mathbf{fd\alpha}$$

Therefore,

$$\overline{\int_a^b} \mathbf{fd\alpha} \geq \overline{\int_a^c} \mathbf{fd\alpha} + \overline{\int_c^b} \mathbf{fd\alpha}$$

Therefore ,

$$\int_a^b \mathbf{fd\alpha} \geq \int_a^c \mathbf{fd\alpha} + \int_c^b \mathbf{fd\alpha} \qquad \text{…..(1)}$$

Similarly using lower sums,

$$\begin{aligned} \mathbf{L(P,f,\alpha)} &= \mathbf{L(P_1,f,\alpha)} + \mathbf{L(P_2,f,\alpha)} \\ &\leq \mathbf{\sup L(P_1,f,\alpha)} + \mathbf{\sup L(P_2,f,\alpha)} \\ &\leq \underline{\int_a^c} \mathbf{fd\alpha} + \underline{\int_c^b} \mathbf{fd\alpha} \end{aligned}$$

Therefore ,

$$\mathbf{\sup L(P,f,\alpha)} \leq \underline{\int_a^c} \mathbf{fd\alpha} + \underline{\int_c^b} \mathbf{fd\alpha}$$

Therefore ,

$$\underline{\int_a^b} \mathbf{fd\alpha} \leq \underline{\int_a^c} \mathbf{fd\alpha} + \underline{\int_c^b} \mathbf{fd\alpha}$$

Therefore ,

$$\int_a^b \mathbf{fd\alpha} \leq \int_a^c \mathbf{fd\alpha} + \int_c^b \mathbf{fd\alpha} \qquad \text{….(2)}$$

From (1) and (2), we get

$$\int_a^b \mathbf{fd\alpha} = \int_a^c \mathbf{fd\alpha} + \int_c^b \mathbf{fd\alpha}\,.$$

Hence the theorem.

Theorem2.5

If $\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, b] and if |f(x)| $\leq$ M on [a, b],

then $\left| \int_a^b \mathbf{fd\alpha} \right| \leq \mathrm{M}[\,\alpha\mathbf{(b)} - \alpha\mathbf{(a)}]$

Proof:

Let P be any partition of [a, b],

Since |f(x)| $\leq$ M on [a, b],

$M_k = \sup\{f(x) / x \in [x_{k-1}, x_k]\}$

$\leq$ M, for all k=1,2….n.

Since $\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, b],

$$\begin{aligned} \int_a^b \mathbf{fd\alpha} = \overline{\int_a^b} \mathbf{fd\alpha} &= \mathbf{\inf U(P,f,\alpha)} \\ &\leq \mathbf{U(P,f,\alpha)} \end{aligned}$$

$$= \sum_{k=1}^{n} \mathbf{M}_k \Delta\alpha_k$$

$$\leq M \sum_{k=1}^{n} \Delta\alpha_k$$

$$= M[\alpha(\mathbf{b}) - \alpha(\mathbf{a})] \qquad \text{....(1)}$$

Replacing f by –f , we get

$$-\int_a^b \mathbf{f}\mathbf{d}\alpha \leq M[\alpha(\mathbf{b}) - \alpha(\mathbf{a})] \qquad \text{....(2)}$$

From (1) and (2),

$$\left| \int_a^b \mathbf{f}\mathbf{d}\alpha \right| \leq M[\alpha(\mathbf{b}) - \alpha(\mathbf{a})]$$

Hence the theorem.

Theorem 2.6

If $\mathbf{f} \in \mathbf{R}(\alpha_1)$ on [a, b] and $\mathbf{f} \in \mathbf{R}(\alpha_2)$ on [a, b],
then $\mathbf{f} \in \mathbf{R}(\alpha_1 + \alpha_2)$ on [a, b] and

$$\int_a^b \mathbf{f}\mathbf{d}(\alpha_1 + \alpha_2) = \int_a^b \mathbf{f}\mathbf{d}\alpha_1 + \int_a^b \mathbf{f}\mathbf{d}\alpha_2$$

Proof:

Let P be any partition of [a, b].

Let $\alpha = \alpha_1 + \alpha_2$

Then $\Delta\alpha_k = (\alpha_1 + \alpha_2)(\mathbf{x}_k) - (\alpha_1 + \alpha_2)(\mathbf{x}_{k-1})$

$$= \alpha_1(\mathbf{x}_k) + \alpha_2(\mathbf{x}_k) - [\alpha_1(\mathbf{x}_{k-1}) + \alpha_2(\mathbf{x}_{k-1})]$$

$$= \alpha_1(\mathbf{x}_k) - \alpha_1(\mathbf{x}_{k-1}) + \alpha_2(\mathbf{x}_k) - \alpha_2(\mathbf{x}_{k-1})$$

$$= \Delta(\alpha_1)_k + \Delta(\alpha_2)_k$$

Consider $\mathbf{U}(\mathbf{P},\mathbf{f},\alpha) = \sum_{k=1}^{n} \mathbf{M}_k \Delta\alpha_k$

$$= \sum_{k=1}^{n} \mathbf{M}_k [\Delta(\alpha_1)_k + \Delta(\alpha_2)_k]$$

$$= \sum_{k=1}^{n} \mathbf{M}_k \ \Delta(\alpha_1)_k + \sum_{k=1}^{n} \mathbf{M}_k \ \Delta(\alpha_2)_k$$

$$= \mathbf{U}(\mathbf{P},\mathbf{f},\alpha_1) + \mathbf{U}(\mathbf{P},\mathbf{f},\alpha_2)$$

$$\leq \inf \mathbf{U}(\mathbf{P},\mathbf{f},\alpha_1) + \inf \mathbf{U}(\mathbf{P},\mathbf{f},\alpha_2)$$

$$= \overline{\int_a^b} \mathbf{f}\mathbf{d}\alpha_1 + \overline{\int_a^b} \mathbf{f}\mathbf{d}\alpha_2$$

Therefore ,

$$\inf \mathbf{U}(\mathbf{P},\mathbf{f},\alpha) \leq \overline{\int_a^b} \mathbf{f}\mathbf{d}\alpha_1 + \overline{\int_a^b} \mathbf{f}\mathbf{d}\alpha_2$$

Therefore,

$$\overline{\int_a^b} \mathbf{f}\mathbf{d}\alpha \leq \overline{\int_a^b} \mathbf{f}\mathbf{d}\alpha_1 + \overline{\int_a^b} \mathbf{f}\mathbf{d}\alpha_2$$

Therefore,

$$\int_a^b \mathbf{f}\mathbf{d}\alpha \leq \int_a^b \mathbf{f}\mathbf{d}\alpha_1 + \int_a^b \mathbf{f}\mathbf{d}\alpha_2 \qquad \text{......(1)}$$

Similarly considering the lower sums , we can prove that

$$\int_a^b \mathbf{f}d\alpha \geq \int_a^b \mathbf{f}d\alpha_1 + \int_a^b \mathbf{f}d\alpha_2 \qquad \ldots\ldots(2)$$

From (1) and (2),

$$\int_a^b \mathbf{f}d\alpha = \int_a^b \mathbf{f}d\alpha_1 + \int_a^b \mathbf{f}d\alpha_2 \,.$$

Therefore,

$$\int_a^b \mathbf{f}d(\alpha_1 + \alpha_2) = \int_a^b \mathbf{f}d\alpha_1 + \int_a^b \mathbf{f}d\alpha_2 \,.$$

Hence the theorem.

Theorem 2.7

If $f \in R(\alpha\}$ on [a, b] and c is a positive constant ,

then $f \in R(c\alpha)\}$ on [a, b] and $\int_a^b fd(c\alpha) = c\int_a^b fd\alpha$

Proof

This follows from theorem 2.6

Theorem2.8

If $\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, b] and $\mathbf{g} \in \mathbf{R}(\alpha)$ on [a, b], then

(a) $\mathbf{f}^2 \in \mathbf{R}(\alpha)$ on [a, b]

(b) $\mathbf{fg} \in \mathbf{R}(\alpha)$ on [a, b]

(c) $|\mathbf{f}| \in \mathbf{R}(\alpha)$ on [a, b] and $\left|\int_a^b \mathbf{f}d\alpha\right| \leq \int_a^b |\mathbf{f}|\, d\alpha$

Proof:

(a) Let P be any partition of [a, b].

Let $M_k(f)$ denotes $\sup\{ f(x)/ x \in [x_{k-1}, x_k]\}$ and

$m_k(f)$ denotes $\inf \{ f(x)/ x \in [x_{k-1}, x_k]\}$

Then $M_k(f^2) = \sup\{ f^2(x)/ x \in [x_{k-1}, x_k]\}$

$= [M_k(|f|)]^2$

$m_k(f^2) = [m_k(|f|)]^2$

Since $\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, b],

f is bounded on [a, b].

Therefore there exists a M>0 such that

$|f(x)| \leq M.$

Then $M_k(f) \leq M$ and

$m_k(f) \leq M$, for all k=1,2,…n.

Also for given $\varepsilon>0$, there exists a partition P of [a, b] such that

$$\mathbf{U(P,f,\alpha) - L(P,f,\alpha)} < \varepsilon$$

Therefore ,

$$\begin{aligned} M_k(f^2)- m_k(f^2) &= [M_k(|f|)]^2- [m_k(|f|)]^2 \\ &= [M_k(|f|)+ m_k(|f|)][M_k(|f|)- m_k(|f|)] \\ &\leq 2M[M_k(|f|)- m_k(|f|)] \end{aligned}$$

Therefore ,

$$\begin{aligned} \mathbf{U(P,f^2,\alpha) - L(P,f^2,\alpha)} &= \sum_{k=1}^{n} \mathbf{[M_k(f^2) - m_k(f^2)]\Delta\alpha_k} \\ &\leq 2M \sum_{k=1}^{n} \mathbf{[M_k(f) - m_k(f\ \ )]\Delta\alpha_k} \end{aligned}$$

$$\leq 2M[\mathbf{U(P,f,\alpha)-L(P,f,\alpha)}]$$
$$< \mathbf{2M(\varepsilon/2M)}$$
$$< \varepsilon .$$

Therefore, $\mathbf{f}^2 \in \mathbf{R(\alpha)}$ on [a, b].

Hence part (a) is proved.

(b) Since $\mathbf{f} \in \mathbf{R(\alpha)}$ on [a, b] and $\mathbf{g} \in \mathbf{R(\alpha)}$ on [a, b],

by theorems 2.1 and 2.2,

$\mathbf{f} + \mathbf{g} \in \mathbf{R(\alpha)}$ on [a, b] , and $\mathbf{f} - \mathbf{g} \in \mathbf{R(\alpha)}$ on [a, b].

Therefore by part (a) proved above ,

$\mathbf{(f+g)}^2 \in \mathbf{R(\alpha)}$ on [a, b] and

$\mathbf{(f-g)}^2 \in \mathbf{R(\alpha)}$ on [a, b].

Therefore again by theorems theorem 2.1 and 2.2,

$(1/4)[\ \mathbf{(f+g)}^2 - \mathbf{(f-g)}^2\,\mathbf{]} \in \mathbf{R(\alpha)}$ on [a, b].

i.e $\mathbf{fg} \in \mathbf{R(\alpha)}$ on [a, b].

Hence part (b) is proved.

(c) Since $\mathbf{f} \in \mathbf{R(\alpha)}$ on [a, b],

for given $\varepsilon > 0$, there exists a partition P of [a, b] such that

$$\mathbf{U(P,f,\alpha)-L(P,f,\alpha)} < \varepsilon$$

Let P be any partition of [a, b].

Since $||f(x)| - |f(y)|| \leq |f(x)-f(y)|$,

$$M_k(|f|) - m_k(|f|) = \sup\{\ |\ |f(x)| - |f(y)|\ |\ / x,y \in [x_{k-1}, x_k]\}$$
$$\leq \sup\{|f(x)-f(y)|\ \ /\ \ x,y \in [x_{k-1}, x_k]\}$$
$$\leq M_k(f)-m_k(f)$$

Therefore ,

$$\mathbf{U(P,|f|,\alpha)-L(P,|f|,\alpha)} \leq \mathbf{U(P,f,\alpha)-L(P,f,\alpha)}$$
$$< \varepsilon .$$

Hence $\mathbf{|f|} \in \mathbf{R(\alpha)}$ on [a, b].

Now for all x, $f(x) \leq |f(x)|$ and

$-f(x) \leq |f(x)|$.

Therefore by applying theorem 2.3, we get

$$\int_a^b \mathbf{f d\alpha} \leq \int_a^b \mathbf{|f|\,d\alpha} \quad \text{and}$$

$$-\int_a^b \mathbf{f d\alpha} \leq \int_a^b \mathbf{|f|\,d\alpha}$$

Therefore $\left|\int_a^b \mathbf{f d\alpha}\right| \leq \int_a^b \mathbf{|f|\,d\alpha}$.

Hence part (c) is proved.

Hence the theorem.

Theorem2.8

Suppose φ is a strictly increasing continuous function that maps an interval [A, B] onto [a, b].
Suppose α is monotonically increasing on[a, b] and $f \in R(\alpha)$ on [a, b].
Define β and g on [A, B] by

$$\beta(y)= \alpha(\varphi(y)), \qquad g(y)=f(\varphi(y)).$$

Then $g \in R(\beta)$ and $\int_A^B g\,d\beta = \int_a^b f\,d\alpha$.

Proof:

To each partition $P=\{x_0,x_1,\ldots\ldots x_n\}$ of [a, b],
there exists a partition $Q=\{y_0,y_1,\ldots.y_n\}$ of [A, B] such that

$$x_i = \varphi(y_i).$$

All partitions of [A, B] can be obtained in this way.
Since $g(y)=f(\varphi(y))$ on [A,B],
the values taken by g on $[y_{i-1}, y_i]$ are the same as those taken by f on $[x_{i-1}, x_i]$.
Therefore,

$$U(Q, g, \beta)=U(P, f, \alpha) \text{ and }$$
$$L(Q, g, \beta)=L(P, f, \alpha). \qquad \ldots.(1)$$

Since $f \in R(\alpha)$ on [a, b],
for given $\varepsilon > 0$,there exists a partition P of [a, b] such that

$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon .$$

Therefore ,

$$U(Q, g, \beta) - L(Q, g, \beta)= U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$

Therefore $g \in R(\beta)$ on [A, B].

Moreover from (1),

$$\inf U(Q, g, \beta)=\inf U(P, f, \alpha)$$

Therefore ,

$$\overline{\int_A^B} g\,d\beta = \overline{\int_a^b} f\,d\alpha .$$

since $f \in R(\alpha)$ on [a, b] and $g \in R(\beta)$ on [A, B],

$$\int_A^B g\,d\beta = \overline{\int_A^B} g\,d\beta \quad \text{and}$$
$$\int_a^b f\,d\alpha = \overline{\int_a^b} f\,d\alpha .$$

Therefore ,

$$\int_A^B g\,d\beta = \int_a^b f\,d\alpha .$$

Hence the theorem.

Check your progress

1. If a<b, $\int_b^a f d\alpha = ?$
2. Suppose f is a bounded valued function on [a, b] and $f^2 \in R$ on [a, b].Does it follow that $f \in R$ on [a, b].

2.3Let us sum up

In this lesson , we have seen that

- Sum of two Riemann Stieltjes integrable functions is also Riemann Stieltjes integrable.
- Scalar product of a Riemann Stieltjes integrable function is also Riemann Stieltjes integrable.
- Modulus of a Riemann Stieltjes integrable function is also Riemann Stieltjes integrable.
- Square of a Riemann Stieltjes integrable function is also Riemann Stieltjes integrable and

- If a function is Riemann Stieltjes integrable on an interval, then it is also Riemann Stieltjes integrable on any of its subinterval.

2.4 Lesson End Activities

1. Show that $_0\int^1 x^2dx = 3/5$ where $\alpha(n) = x^3$
2. Show that $_0\int^2 [x]dx = 3/5$ where $\alpha(x) = x^2 = 3$.

2.5 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

LESSON – 3

INTEGRATION AND DIFFERENTIATION

3.0Introduction

In this lesson, we are going to see that differentiation and integration, are in a certain sense, inverse operations.

3.1Aims and objectives

After studying this lesson, you would know

- the fundamental theorems on calculus and
- the theorem on integration by parts.

3.2Integration and differentiation

Theorem 3.1

Let $f \in R$ on [a, b](i.e f is Riemann-integrable on [a, b]).

For $a \le x \le b$, define $F(x)=\int_a^x f(t)dt$.

Then F is continuous on [a, b];
Furthermore, if f is continuous at a point x_0 of [a, b],
then F is differentiable at x_0 and

$$F'(x_0)=f(x_0).$$

Proof:
Since $f \in R$ on [a, b],
f is bounded.
Therefore there exists M>0 such that

$$|f(t)| \le M \quad \text{for } a \le t \le b.$$

Therefore if $a \le x \le y \le b$, then

$$\begin{aligned} |F(y) - F(x)| &= \left|\int_a^y f(t)dt - \int_a^x f(t)dt\right| \\ &= \left|\int_x^y f(t)dt\right| \\ &\le \int_x^y |f(t)|\,dt \\ &\le M\int_x^y dt = M(y-x) \end{aligned}$$

i.e $|F(y) - F(x)| \le M(y-x)$.
Therefore,

$| F(y) - F(x) | < \varepsilon$ provided that $|y-x| < \varepsilon/M$.

Hence F is continuous on [a, b].

Suppose if f is continuous at x_0.

then for given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$| f(t)-f(x_0)| < \varepsilon$ whenever $|t-x_0| < \delta$.

Hence if $x_0-\delta < s \leq x_0 \leq t < x_0+\delta$ and $a \leq s < t \leq b$,

$$\left| \frac{F(t)-F(s)}{t-s} - f(x_0) \right| = \left| \frac{1}{t-s}\int_s^t f(t)dt - f(x_0) \right|$$

$$= \left| \frac{1}{t-s}\int_s^t [f(t)-f(x_0)]dt \right|$$

$$\leq \frac{1}{t-s}\int_s^t |f(t)-f(x_0)|\, dt$$

$$< \frac{\varepsilon}{t-s}\int_s^t dt$$

$$< \frac{\varepsilon}{t-s}[t-s]$$

$$< \varepsilon$$

Therefore ,

$$\left| \frac{F(t)-F(s)}{t-s} - f(x_0) \right| < \varepsilon \qquad \text{whenever } x_0-\delta < s \leq x_0 \leq t < x_0+\delta$$

Therefore $F'(x_0)=f(x_0)$.

Hence the theorem.

Theorem 3.2 Fundamental theorem of calculus.

If $f \in R$ on [a, b] and if there exists a differentiable function F on [a. b] such that $F'=f$, then

$$\int_a^b f(x)dx = F(b)-F(a).$$

Proof:

Let $\varepsilon > 0$ be given.

Since $f \in R$ on [a, b],

there exists a partition $P = \{x_0, x_1, x_2, \ldots\ldots x_n\}$ of [a, b] such that

$$U(P,f) - L(P,f) < \varepsilon.$$

By the Mean value theorem , there exist points t_i such that

$$F(x_i) - F(x_{i-1}) = f(t_i)[x_i - x_{i-1}]$$

$$= f(t_i)\Delta x_i \qquad \textbf{for } i=1,2,\ldots.n.$$

Therefore,

$$\sum_{i=1}^{n} f(t_i)\Delta x_i = \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})]$$

$$= F(x_n) - F(x_0)$$

$$= F(b) - F(a).$$

Also $L(P,f) \leq \sum_{i=1}^{n} f(t_i)\Delta x_i \leq U(P,f)$

And $$\mathbf{L(P,f)} \le \int_a^b \mathbf{f(x)dx} \le \mathbf{U(P,f)}$$

Therefore,

$$\left| \int_a^b \mathbf{f(x)dx} - [\mathbf{F(b)-F(a)}] \right| = \left| \int_a^b \mathbf{f(x)dx} - \sum_{i=1}^{n} \mathbf{f(t_i)\Delta x_i} \right|$$
$$\le \mathbf{U(P,f)-L(P,f)}$$
$$< \varepsilon .$$

Since ε was arbitrary ,

$$\int_a^b \mathbf{f(x)dx} = F(b)-F(a).$$

Hence the theorem.

Theorem3.3 Integration by parts.

Suppose F and G are differentiable functions on [a, b], $F' = f \in R$ and $G' = g \in R$.

Then $\int_a^b \mathbf{F(x)g(x)dx} = \mathbf{F(b)G(b) - F(a)G(a)} - \int_a^b \mathbf{f(x)G(x)dx}$.

Proof:

Let $\mathbf{H(x) = F(x)G(x)}$.

Then $$\mathbf{H' = (FG)'}$$
$$= \mathbf{FG'+F'G}$$
$$= \mathbf{Fg+fG} \in \mathbf{R} .$$

Applying theorem3.2 to the differentiable function H,

$$\int_a^b [\mathbf{F(x)g(x) + f(x)G(x)}]\mathbf{dx} = \mathbf{H(b)-H(a).}$$

Therefore,

$$\int_a^b \mathbf{F(x)g(x)dx} + \int_a^b \mathbf{f(x)G(x)dx} = \mathbf{F(b)G(b)-F(a)G(a).}$$

Therefore,

$$\int_a^b \mathbf{F(x)g(x)dx} = \mathbf{F(b)G(b) - F(a)G(a)} - \int_a^b \mathbf{f(x)G(x)dx} .$$

Hence the theorem.

Check your progress

Derive the identities

1. $\sum_{k=1}^{n} \frac{1}{k^s} = \frac{1}{n^{s-1}} + s\int_1^n \frac{[x]}{x^{s+1}} dx$,if s 1
2. $\sum_{k=1}^{n} \frac{1}{k} = \log n - \int_1^n \frac{x-[x]}{x^2} dx + 1$

3.3Let us sum up

In this lesson, you have studied the important fundamental theorems on calculus that

- if F is the integral of f with respect to α on [a,b], then differentiation of F at a point gives f provided f is continuous at that point.
- If F is the integral of f with respect to α on [a,b], then $\int_a^b \mathbf{f(x)dx} = F(b)-F(a)$ and
- the result similar to integration by parts you have studied in School.

3.4 Lesson End Activities

If it is a real, Continosly differentiable function on [a, b] with f(a) = f(b) =0 and $_a\int^b f^2(x)\,dx = 1$, prove that $_a\int^b x\, f(x)\, f'(x)\,dx = -\frac{1}{2}$.

3.5 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

LESSON – 4

INTEGRATION OF VECTOR VALUED FUNCTIONS.

4.0 Introduction

In this lesson , we are going to see how to integrate vector valued functions and whether the theorems on integration we have already seen are valid for these functions.

4.1 Aim and objectives

After studying this lesson you would know

- The definition of integration of vector valued functions
- Sum of the integrals of two vector valued functions is the sum of the integrals
- Integral of a scalar multiple of a vector valued function is the scalar multiple of the integral and
- Most of the results that are true for real functions is true for vector valued functions also.

4.2 Integration of vector valued functions.

Definition:

Let $f_1, f_2, \ldots\ldots f_k$ be real valued functions on [a, b].

Let **f=(** $f_1, f_2, \ldots\ldots f_k$)be the corresponding mapping of [a, b] onto R^k.

If α increases monotonically on [a, b] and if f_j **ЄR(**α) for j=1,2,…k,

then we say that **f ЄR(**α} on [a, b] and define $\int_a^b f d\alpha$ as

$$\int_a^b f d\alpha = \left(\int_a^b f_1 d\alpha , \int_a^b f_2 d\alpha ,\ldots\ldots \int_a^b f_k d\alpha \right)$$

i.e $\int_a^b f d\alpha$ is the point in R^k whose jth coordinate is $\int_a^b f_j d\alpha$.

Let **f=(** $f_1, f_2, \ldots\ldots f_k$) and **g=(** $g_1, g_2, \ldots\ldots g_k$) be vector valued functions on [a, b].

Then by the way in which we have defined $\int_a^b f d\alpha$ and using the results we have seen in Lesson 2 and 3, we get the following results.

Theorem 4.1

If **f** $\in$ **R(**α**)** and **g** $\in$ **R(**α**)** on [a, b], then **f+g** $\in$ **R(**α**)** and

$$\int_a^b (f+g) d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha ,$$

Proof:

Since **f** $\in$ **R(**α**)** and **g** $\in$ **R(**α**)** on [a, b],

$f_j,\ g_j \in \mathbf{R}(\alpha)$ on [a, b], for j=1,2,…k.

Hence $f_j+g_j \in \mathbf{R}(\alpha)$ on [a, b], for j=1,2,…k.

Therefore,

$(\mathbf{f+g}) \in \mathbf{R}(\alpha)$ on [a, b]

And $\int_a^b (f+g)d\alpha = (\int_a^b (f_1+g_1)d\alpha\ ,\ \int_a^b (f_2+g_2)d\alpha\ ,\dots \int_a^b (f_k+g_k)d\alpha\ )$

$$=(\int_a^b f_1 d\alpha + \int_a^b g_1 d\alpha\ ,\ \int_a^b f_2 d\alpha + \int_a^b g_2 d\alpha\ ,\dots\dots \int_a^b f_k d\alpha + \int_a^b g_k d\alpha\ )$$

$$=(\int_a^b f_1 d\alpha\ ,\ \int_a^b f_2 d\alpha\ ,\dots \int_a^b f_k d\alpha\ ) + (\int_a^b g_1 d\alpha\ ,\ \int_a^b g_2 d\alpha\ ,\dots \int_a^b g_k d\alpha\ )$$

$$= \int_a^b f d\alpha + \int_a^b g d\alpha$$

Hence the theorem.

In the similar way we can prove the following results.

Theorem4.2

If $\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, b] , then $\mathbf{c\,f} \in \mathbf{R}(\alpha)$ on [a, b], for any constant c and

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{cf}d\alpha = \mathbf{c}\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f}d\alpha\ .$$

Theorem4.3

If $\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, b] and if a<c<b, then $\mathbf{f} \in \mathbf{R}(\alpha)$ on [a, c] and on [c, b], and

$$\int_{\mathbf{a}}^{\mathbf{c}} \mathbf{f}d\alpha + \int_{\mathbf{c}}^{\mathbf{b}} \mathbf{f}d\alpha = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f}d\alpha\ .$$

Theorem4.4

If $\mathbf{f} \in \mathbf{R}(\alpha_1)$ on [a, b] and $\mathbf{f} \in \mathbf{R}(\alpha_2)$ on [a, b], then $\mathbf{f} \in \mathbf{R}(\alpha_1+\alpha_2)$ on [a, b] and

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f}\mathbf{d}(\alpha_1+\alpha_2) = \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f}\mathbf{d}\alpha_1 + \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f}\mathbf{d}\alpha_2$$

Theorem4.5

If $f \in R(\alpha\}$ on [a, b] and c is a positive constant , then $f \in R(c\alpha)\}$ on [a, b] and

$$\int_a^b f d(c\alpha) = c\int_a^b f d\alpha$$

Theorem4.6

Let $f \in R$ on [a, b](i.e f is Riemann-integrable on [a, b]).

For $a \le x \le b$, define F(x)= $\int_{\mathbf{a}}^{\mathbf{x}} \mathbf{f(t)dt}$.

Then F is continuous on [a, b];

Furthermore, if f is continuous at a point x_0 of [a, b],

then F is differentiable at x_0 and $F'(x_0)=f(x_0)$.

Theorem4.7

If f and **F** maps [a, b] onto R^k,if $f \in R$ on [a, b] and if **F'=f**, then

$$\int_a^b f(t)dt = \mathbf{F}(b)-\mathbf{F}(a).$$

Theorem4.8

If **f** maps [a, b] onto R^k, and
if $\mathbf{f} \in \mathbf{R}(\alpha)$ for some monotonically increasing function α on [a, b] ,
Then $|\mathbf{f}| \in \mathbf{R}(\alpha)$ on [a, b] and

$$\left|\int_a^b \mathbf{f} d\alpha\right| \leq \int_a^b |\mathbf{f}|\, d\alpha .$$

Proof
If $f_1, f_2, \ldots f_k$ are the components of **f** ,then

$$|\mathbf{f}| = (f_1^2+f_2^2+\ldots+f_k^2)^{1/2} .$$

Since $\mathbf{f} \in \mathbf{R}(\alpha\}$ on [a, b],
By definition,

each $f_j \in \mathbf{R}(\alpha)$ for j=1,2,…k and

By theorem2.8,

$$f_j^2 \in \mathbf{R}(\alpha\} \text{ for } j=1,2,\ldots k .$$

Hence by theorem 2.1,

$$f_1^2+f_2^2+\ldots+f_k^2 \in \mathbf{R}(\alpha).$$

Hence

$$|\mathbf{f}|=(f_1^2+f_2^2+\ldots+f_k^2)^{1/2} \in \mathbf{R}(\alpha).$$

[Since square root of a continuous function is continuous on [0, M] , for every real M]

To prove that $\left|\int_a^b \mathbf{f} d\alpha\right| \leq \int_a^b |\mathbf{f}|\, d\alpha .$

Let $\mathbf{y}=(y_1, y_2, \ldots y_k)$, where $y_j= \int_a^b f_j d\alpha$

Then $\mathbf{y}= \int_a^b f d\alpha$.

And $|\mathbf{y}|^2 = \sum_{j=1}^{k} y_j^2$

$$= \sum_{j=1}^{k} y_j \int_a^b f_j d\alpha$$

$$= \int_a^b \left(\sum_{j=1}^{n} y_j f_j\right) d\alpha$$

From Schwarz inequality,

$$\sum_{j=1}^{k} y_j f_j \quad |\mathbf{y}||\mathbf{f}(t)| \qquad (a \quad t \quad b)$$

Therefore

$$|\mathbf{y}|^2 \quad |\mathbf{y}| \int_a^b | f \,| d\alpha$$

Therefore if **y** 0, dividing this enequality by $|\mathbf{y}|$, we get

$$|\mathbf{y}| \quad \int_a^b | f \,| d\alpha \,\textbf{.}$$

Hence

$$\left|\int_a^b \mathbf{f} d\alpha\right| \leq \int_a^b |\mathbf{f}|\, d\alpha .$$

Hence the theorem.

4.3 Let us sum up

From what we have seen ,it is clear that most of the properties of integrals of real or complex valued functions are valid for vector valued functions also.

4.4 Lesson End Activity

if $f : [0, 1] \mid\rightarrow R^3$ is defined by $f(t) = (t, t^2, t^3)$ & if $\propto(t) = t \ \forall \ t \ t \ [0, 1]$, find ${}_0\int^1 f \, dx$

4.5 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

Lesson - 5

RECTIFIABLE CURVES

5.0Introduction

In this lesson you are going to study about rectifiable curves.

5.1Aims and objectives

After studying this lesson you would know

- What is a rectifiable curve
- Length of a rectifiable curve and that
- The length is given by a Riemann integral.

5.2Rectifiable curves

Definition:

A continuous mapping γ of an interval [a, b] into R^k is called a curve in R^k. or γ is a curve on [a, b].

If γ is one-to-one, γ is called an arc.

If $\gamma(a) = \gamma(b)$, γ is called a closed curve.

Definition:

To each partition P= $\{x_0, x_1, x_2, \ldots.. x_n\}$ of [a, b] and to each curve γ on [a, b] ,

we associate a number $\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$

where $|\gamma(x_i) - \gamma(x_{i-1})|$ = distance between the points $\gamma(x_{i-1})$ and $\gamma(x_i)$

$\Lambda(P, \gamma)$ = length of a polygonal path with vertices at $\gamma(x_0)$, $\gamma(x_1)$,…. $\gamma(xn)$.

As the partition P becomes finer and finer,

the polygon approaches γ more and more closely.

The length of Λ is defined as

$$\Lambda(\gamma) = \sup \Lambda(P, \gamma),$$

where the supremum is taken over all partitions of [a, b].

If $\Lambda(\gamma) < \infty$, γ is said to be rectifiable.

Theorem 5.1

If γ' is continuous on [a, b], then $\leq$ is rectifiable, and $\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt$.

Proof:

If $a \le x_{i-1} < x_i \le b$, then

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left| \int_{x_{i-1}}^{x_i} \gamma'(t)dt \right|$$

$$\le \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt$$

Hence $\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$

$$\le \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\gamma'(t)| dt$$

$$\le \int_a^b |\gamma'(t)|\, dt$$

for every partition P of [a,b].

Therefore ,

$$\Lambda(\gamma) \le \int_a^b |\gamma'(t)|\, dt \qquad \ldots.(1)$$

To prove the opposite inequality,

let $\varepsilon > 0$ be given.

Since γ' is uniformly continuous on [a, b], there exists a $\delta > 0$ such that

$$|\gamma'(s) - \gamma'(t)| < \varepsilon \quad \text{whenever } |s - t| < \delta.$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of [a, b], with $\Delta x_i < \delta$ for all i.

Therefore if $x_{i-1} \le t \le x_i$,

$$|\gamma'(t) - \gamma'(x_i)| < \varepsilon$$

Therefore,

$$|\gamma'(t)| - |\gamma'(x_i)| \le |\gamma'(t) - \gamma'(x_i)|$$

$$< \varepsilon$$

Therefore,

$$|\gamma'(t)| \le |\gamma'(x_i)| + \varepsilon.$$

Hence $\int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt \le \int_{x_{i-1}}^{x_i} (|\gamma'(x_i)| + \varepsilon)dt$

$$\le |\gamma'(t)| \Delta x_i + \varepsilon\, \Delta x_i$$

$$\le \left| \int_{x_{i-1}}^{x_i} \gamma'(x_i)dt \right| + \varepsilon\, \Delta x_i$$

$$\le \left| \int_{x_{i-1}}^{x_i} [\gamma'(t) + \gamma'(x_i) - \gamma'(t)]dt \right. + \varepsilon\, \Delta x_i$$

$$\le \left| \int_{x_{i-1}}^{x_i} \gamma'(t)dt \right| + \left| \int_{x_{i-1}}^{x_i} [\gamma'(x_i) - \gamma'(t)]dt \right| + \varepsilon\, \Delta x_i$$

$$\le |\gamma(x_i) - \gamma(x_{i-1})| + \varepsilon\ \Delta x_i ++ \varepsilon\, \Delta x_i$$

$$\le |\gamma(x_i) - \gamma(x_{i-1})| + 2\varepsilon\, \Delta x_i$$

Therefore ,

$$\int_a^b |\gamma'(t)|\, dt = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |\gamma'(t)| dt$$

$$\le \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})| + 2\varepsilon \sum_{i=1}^{n} \Delta x_i$$

$$\leq \Lambda(P, \gamma) + 2\varepsilon(\mathbf{b}-\mathbf{a})$$
$$\leq \Lambda(\gamma) + 2\varepsilon(\mathbf{b}-\mathbf{a}).$$

Since $\varepsilon > 0$ was arbitrary,

$$\int_a^b |\gamma'(t)|\, dt \leq \Lambda(\gamma) \qquad \ldots(2).$$

From (1) and (2), we get

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt$$

Hence the theorem

Check your progress

Let $\gamma_1, \gamma_2, \gamma_3$ be curves in the complex plane , defined on $[0, 2\pi]$ by $\gamma_1(t)=e^{it}$, $\gamma_2(t)=e^{2it}$, $\gamma_3(t)=e^{2\pi it\sin(1/t)}$.

Show that

1. These three curves have the same range
2. γ_1 and γ_2 are rectifiable.
3. length of γ_1 is 2π.
4. length of γ_2 is 4π and
5. γ_3 is not rectifiable.

5.3 Let us sum up

In this lesson you have studied

- the definition of a rectifiable curve,
- its length and that
- its length is equal to the Riemann integral of the modulus of the derivative of the rectifiable curve.

5.4 Lesson End Activities

if γ_1 γ_2 are curves in the complex plane defined on [0, 1] & [0, 2] respectively by $\gamma_1(t) = e^{2\Pi it}$; $\gamma_{2}(t) = e^{2\Pi it}$; $t \in [0,1]$, Prove that length of γ_2 is twice that of γ_1 .

5.5 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

UNIT II

SEQUENCES AND SERIES OF FUNCTIONS

LESSON – 6

UNIFORM CONVERGENCE AND CONTINUITY

6.0 Introduction

In this lesson, we are going to see about uniform convergence of a sequence of functions and continuity of the limit function.

6.1 Aims and objectives

After studying this unit, you should be able to identify

- pointwise convergence of a sequence of functions
- uniform convergence of a sequence of functions
- the limit function
- continuity of the limit function

6.2Poitwise convergence and Uniform convergence

Definition: pointwise convergence of a sequence of functions

Suppose $\{f_n\}$, n=1,2,….. is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for each x in E.
Define the function f by

$$f(x)=\lim_{n\to\infty} f_n(x) \qquad \text{for each } x\in E$$

Then we say that $\{f_n\}$ converges pointwise to f on E and
f is called the limit function.

Definition: pointwise convergence of a series of functions

Suppose $\{f_n\}$, n=1,2,….. is a sequence of functions defined on a set E, and suppose that the series $\sum_{n=1}^{\infty} f_n(x)$ converges for each x in E

$$\text{and } f(x)= \sum_{n=1}^{\infty} f_n(x)\,, \qquad \text{for each } x\in E,$$

Then f is called the sum of the series $\sum f_n$.

Examples

1. Let $f_n(x)=x^n$, 0 x 1, for n=1,2,3,…

Then $\lim_{n›}$ $f_n(x)=\lim_{n›}$ x^n =0, if 0 x<1

$=1,\qquad$ if $x=1$.

Hence the limit function f is

$f(x)=0,\quad$ if $0\ x<1$

$=1,\quad$ if $x=1$.

2.Let $f_n(x)=\dfrac{x^2}{(1+x^2)^n}$ $\qquad$ x real, n=1,2,3,….

Then $\displaystyle\sum_{n=0}^{\infty} f_n(x)=\sum_{n=0}^{\infty}\frac{x^2}{(1+x^2)^n}$

$=x^2[1-\dfrac{1}{1+x^2}]^{-1}$

$=x^2[\dfrac{1+x^2-1}{1+x^2}]^{-1}$

$=1+x^2\quad$ if $x\ 0$.

If x=0, $\displaystyle\sum_{n=0}^{\infty} f_n(0)=0$

Hence sum of the series is

$f(x)=0\ $ if $x=0$

$=1\ $ if $x\ 0$

Definition:Uniform convergence of a sequence of functions

A sequence of functions $\{f_n\}$, n=1,2,3,… is said to converge uniformly on E to a function f , if for every $\varepsilon>0$, there exists an integer N such that $n\ N$ implies

$|f_n(x)-f(x)|\ \varepsilon\quad$ for all $x\in E$.

we write this as f_n›f uniformly on E.

Remark:

Here N depends only on ε and not on x.

Hence the convergence is said to be uniform.

Otherwise the convergence is only pointwise convergence.

Theorem6.2.1 Cauchy criterion for uniform onvergence

The sequence of functions $\{f_n\}$, defined on E converges uniformly on E if and only if there exists an integer N such that $m\ N,\ n\ N,\ x\in E$ implies

$|f_n(x)-f_m(x)|\ \varepsilon$.

Proof

Suppose that $\{f_n\}$ converges uniformly on E to a limit function f.

Then by definition of uniform convergence,

For given $\varepsilon>0$, there exists an N such that $n\ N$ implies

$|f_n(x)-f(x)|\ \varepsilon/2\qquad$ for all $x\in E$ $\qquad$ ...(1)

Therefore if $m\ N,\ n\ N,\ x\in E$ then

$|f_n(x)-f_m(x)|=|(f_n(x)-f(x)) - (f_m(x)-f(x))|$

$=|f_n(x)-f(x)| + |f_m(x)-f(x)|$

$\varepsilon/2 + \varepsilon/2$

ε

Conversely ,

assume that for given ε>0, there exists an integer N such that that m N, n N, $x \in E$ implies

$|f_n(x)-f_m(x)|\ \ \varepsilon$. ...(2)

Then for every x in E,

$\{f_n(x)\}$ is a Cauchy sequence of numbers.

Since a Cauchy sequence of real or complex numbers converges,

$\{f_n(x)\}$ converges for every x in E.

Define f on E as

$f(x)=\lim_{n›}\ f_n(x)$ for each x in E.

Then $\{f_n\}$ converges pointwise on E.

To show that the convergence is uniform.

In (2), let n be fixed and let m› ,

Then $|f_n(x)-f(x)|\ \ \varepsilon$ for all x in E and for all n N.

Hence the theorem.

Theorem 6.2.2

Suppose $\lim_{n›}\ f_n(x) = f(x)$.

Put $M_n = \sup\{|f_n(x)-f(x)|,\ x \in E\}$

Then f_n›f uniformly on E if and only if M_n›0 as n› .

Proof

By definition f_n›f uniformly on E

if and only if ,

for given ε>0, there exists N such that n N implies

$|f_n(x)-f(x)|\ \ \varepsilon$ for all $x \in E$.

if and only if

$M_n=\sup\{|f_n(x)-f(x)|,\ x \in E\}\ \ \varepsilon$

if and only if

$\lim_{n›}\ M_n=0$.

Hence the theorem.

Theorem6.2.3

Suppose $\{f_n\}$ is a sequence of functions defined on E, and suppose that $|f_n(x)|\ M_n$ ($x \in E$, n=1,2,3,….).

Then $\sum f_n$ converges uniformly on E if $\sum M_n$ converges.

Proof

Let $t_n=M_1+M_2+\ldots M_n$

And $s_n(x)=\sum_{k=1}^{n} f_k(x)$

Assume that $\sum M_n$ converges.

Hence $\{t_n\}$ is a Cauchy sequence.

Hence , for given ε>0, there exists N such that m N, n N implies

$|t_m-t_n|\ \ \varepsilon$ ….(1)

Let m>n N

Then (1) implies that

$$\sum_{k=n+1}^{m} M_k \quad \varepsilon \qquad \ldots(2)$$

Therefore for m>n N ,

$$|s_m(x)-s_n(x)| = |\sum_{k=n+1}^{m} f_k(x)|$$

$$\sum_{k=1}^{n} | f_k(x)|$$

$$\sum_{k=n+1}^{m} M_k$$

ε for all x in E.

Hence $\{s_n\}$ satisfies the Cauchy criterion for uniform convergence.
Hence $\{s_n\}$ converges uniformly on E.
Hence $\sum f_n$ converges uniformly on E.
Hence the theorem.

6.3 Uniform convergence and continuity

Definition: Continuity of a function

A function f defined on E is said to be continuous at x ∈ E if

$\lim_{t› x} f(t)=f(x)$.

Theorem6.3.1

Suppose $f_n›f$ uniformly on a set E in a metric space.
Let x be a limit point of E, and suppose that

$\lim_{t› x} f_n(t)=A_n$ (n=1,2,3,…) …(1)

Then $\{A_n\}$ converges and $\lim_{t› x} f(t)=\lim_{n›} A_n$. . ..(2)

In other words, $\lim_{t› x} \lim_{n›} f_n(t)=\lim_{n›} \lim_{t› x} f_n(t)$ …(3)

Proof

Since $f_n›f$ uniformly on E, by definition,
for given ε>0,
there exists an integer N_1 such that n N_1 implies

$|f_n(t)- f(t)|$ ε/3 for all t ∈ E …(4)

By cauchy's criterion for uniform convergence,
for the same ε>0,
there exists an integer N_2 such that m N_2, n N_2 ,t∈ E implies that

$|f_n(t)- f_m(t)|$ ε/3 for all t ∈ E …(5)

Letting t›x in(5), we get

$|A_n-A_m|$ ε/3 for m N_2, n N_2 …(6)

Therefore $\{A_n\}$ is a Cauchy sequence.
Hence $\{A_n\}$ converges, say to A.
Therefore for the same ε>0 taken above,
there exists an integer N_3 such that n N_3 implies

$|A_n-A|$ ε …(7)

Let $N=\max(N_1,N_2,N_3)$
Then from(4),

$|f_N(t)- f(t)|$ ε/3 …(8)

From(7), $|A_N-A|$ ε/3 …(9)

By assumption(1),

$\lim_{t› x} f_N(t) = A_N$.

Therefore there exists a neighborhood V of x such that

$|f_N(t)-A_N|$ ε/3 for all t ∈ V E, t x …(10)

From(8),(9)and (10),

$|f(t)-A|=|f(t) – f_N(t) + f_N(t) - A_N + A_N - A |$
$| f(t) – f_N(t)| + |f_N(t) - A_N| + |A_N - A |$
$ε/3+ ε/3+ ε/3$
$ε$ for all $t ∈V\ \ E$ and $t\ \ x$,

Hence $\lim_{t› x} f(t)=A$.
i.e $\lim_{t› x} f(t)= \lim_{n›}\ A_n$.
i.e $\lim_{t› x} \lim_{n›}\ f_n(t)=\lim_{n›}\ \lim_{t› x} f_n(t)$
Hence the theorem.

Remark:
The above theorem says that if $f_n›f$ uniformly and x is a limit point of E, then the limit processes can be interchanged.

Theorem6.3.2

If $\{f_n\}$ is a sequence of continuous functions on E and if $f_n›f$ uniformly on E, then f is continuous.

Proof

Let $\{f_n\}$ be sequence of continuous functions on E.
Let x be any point of E.
Then each f_n is continuous at x.
Therefore,

$$\lim_{t› x} f_n(t)=f_n(x), \qquad n=1,2,3,…$$

Applying the above theorem6.3.1 and replacing A_n by $f_n(x)$, we get

$$\lim_{t› x} f(t)=\lim_{n›}\ f_n(x).$$

Since $f_n›f$ on E,

$$\lim_{n›}\ f_n(x)=f(x).$$

Hence $\lim_{t› x} f(t)= f(x)$.
Hence the theorem.

Theorem6.3.3

Suppose K is compact and
(a) $\{f_n\}$ is a sequence of continuous functions on K
(b) $\{f_n\}$ converges pointwise to a continuous function f on K,
(c) $f_n(x)\ \ f_{n+1}(x)$ for all $x∈E$, $n=1,2,3,…$
Then $f_n›f$ uniformly on K.

Proof

Let $g_n=f_n-f$, $n=1,2,3,….$
Since f_n and f are continuous functions on K,
g_n is also continuous on K, $n=1,2,3,…$
Since $f_n ›f$ pointwise on K,
$g_n›0$ pointwise on K.
Since $f_n(x)\ \ f_{n+1}(x)$ for all $x∈K$, $n=1,2,3,..$
$f_n(x) – f(x)\ \ \ f_{n+1}(x) – f(x)$ for all $x∈K$, $n=1,2,3,..$
Hence $g_n(x)\ \ \ g_{n+1}(x)$, for all $x∈K$, $n=1,2,3,..$ …(1)

Claim: $g_n›0$ uniformly on K.
Let $ε> 0$ be given.
Let $K_n = \{x∈K /\ \ g_n(x)\ \ ε\}$
$=g_n^{-1}([ε,\ \ ))$ …(2)

Since each g_n is continuous and inverse image of a closed set under a continuous function is closed,

each K_n is a closed subset of K.

Since K is compact and a closed subset of a compact set is closed,

each K_n is compact.

If $x \in K_{n+1},$ then $g_{n+1}(x)\ \ \varepsilon$

Therefore,

$g_n(x)\ \ g_{n+1}(x)\ \ \varepsilon$

Hence $x \in K_n$

Hence $K_{n+1} \subseteq K_n$

Fix $x \in K.$

Since $g_n(x)$› 0 ,

for given ε>0, there exists an integer N_1 such that n N_1 implies

$g_n(x)\ \ \varepsilon$

This implies that

$x \notin K_n$ for all n N_1.

This implies that

K_n is empty.

Hence there exists an integer N such that K_N is empty.

This implies that

$g_N(x)\ \ \varepsilon$ for all $x \in K.$

From (1), it is clear that

$\{g_n\}$ is monotonically decreasing sequence.

Therefore,

$g_n(x)\ g_N(x)$ for all n N.

Hence 0 $g_n(x)\ \ g_N(x)\ \ \varepsilon$ for all $x \in K$, n N

Hence,

g_n›0 uniformly on K.

Hence,

$f_n - f$ ›0 uniformly on K.

Hence f_n › f uniformly on K.

Hence the theorem.

Check your progress

1. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n+g_n\}$ converge uniformly on E.

If in addition, $\{f_n\}$ and $\{g_n\}$ are bounded, prove that $\{f_n g_n\}$ converges uniformly on E.

2. Is the converse of the theorem6.3.2 true?
3. Is the compactness necessary in theorem6.3.3?

Answer:

2. No.

A sequence of continuous functions may converge to a continuous function although the convergence is not uniform.

Example:

Let $f_n(x)= n^2x(1-x^2)^{n,}$ 0 x 1, n=1,2,3,…

Each f_n is a continuous function on [0, 1].

And $\lim_{n›} f_n(x)=0$ 0<x 1

Since $f_n(0)=0$, for all n,

$\lim_{n›} f_n(x)=0$ for x=0

Hence the limit function is

$$f(x)=0 \qquad 0 \quad x \quad 1.$$

Here $f(x)$ is also a continuous function on [0, 1].
But $\{f_n\}$ does not converge uniformly to f on [0, 1].

3. Yes.
Example:

Let $f_n(x) = \dfrac{1}{nx+1}$ $\qquad 0<x<1, n=1,2,3,\ldots$

Then $f_n(x)$›0 monotonically in (0, 1),
But the convergence is not uniform.

6.4Let us sum up

In this lesson , we have seen

- Pointwise convergence of a sequence of functions.
- Uniform convergence of a sequence of functions.
- Pointwise convergence of a series of functions.
- Uniform convergence of a series of functions.

We have also seen that

- Limit of a sequence of continuous functions converging uniformly is continuous.
- If a sequence of continuous functions is monotonic and converges pointwise to a continuous function on a compact set, then the convergence is uniform.

6.5 Lesson End Activities

1. Prove that the sequence of functions $\{f_n\}^{\propto}_{n-1}$ converges pointwise but not uniformly. Given, $fn(x) = n/ (1+n^2x^2)$; $x \in [0, 1]$.
2. Show that the sequence $\{f_n\}^{\propto}_{n-1}$ where $f_n(x) = \sin nx/ \sqrt{n}$ is uniformly conversent on $[0, \prod]$.

6.6 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

Lesson 7 Uniform convergence and integration

7.0 Introduction

In this lesson, we are going to learn about the integrability of the limit function of a sequence of integrable functions.

7.1 Aims and objectives

After studying this lesson, you would know

- whether the limit function of a sequence of integrable functions is integrable and
- if it is so, can the operations of integration and limit can be interchanged.

7.2 Uniform convergence and integration

Theorem 7.1

Let α be a monotonically decreasing function on [a, b].
Suppose $f_n \in R(\alpha)$ on [a, b], for n=1,2,3,.. and
suppose $f_n›f$ uniformly on [a, b],
then $f \in R(\alpha)$ on [a, b] and

$$\int_a^b f d\alpha \ = \lim_{n›} \int_a^b f_n d\alpha$$

Proof

Let $\{f_n\}$ be a sequence of real valued functions on [a, b] such that
(1) $f_n \in R(\alpha)$ on [a, b], for n=1,2,3,.. and
(2) $f_n›f$ uniformly on [a, b]

Let $\varepsilon_n = \sup_{a \ x \ b} |f_n(x)-f(x)|$
Then $\varepsilon_n›0$ as $n›$.
This implies that

$$|f_n(x)-f(x)| \quad \varepsilon_n \qquad a \ x \ b$$

(or) $$|f(x)-f_n(x)| \,| \quad \varepsilon_n \qquad a \ x \ b$$

(i.e) $$-\varepsilon_n \quad f(x)-f_n(x) \quad \varepsilon \qquad a \ x \ b$$

Therefore,

$$f_n(x)-\varepsilon \quad f(x) \quad f_n(x)+\varepsilon_n \qquad a \ x \ b \qquad \ldots(1)$$

Hence ,

$$f_n(x)-\varepsilon \quad f(x) \qquad a \ x \ b$$

Hence ,

$$\underline{\int_a^b} (f_n - \varepsilon_n) d\alpha \qquad \underline{\int_a^b} f d\alpha \qquad \ldots(2)$$

Similarly from (1),

$$f(x) \quad f_n(x)+\varepsilon_n \qquad a \ x \ b$$

Hence,

$$\overline{\int_a^b} f d\alpha \qquad \overline{\int_a^b} (f_n + \varepsilon_n) d\alpha \qquad \ldots(3)$$

Since $f_n \in R(\alpha)$ on [a, b], n=1,2,3,..
$f_n - \varepsilon_n \in R(\alpha)$ on [a, b] and

$f_n + \varepsilon_n \in R(\alpha)$ on [a, b]

Hence ,

$$\underline{\int_a^b}(f_n - \varepsilon_n)d\alpha = \int_a^b (f_n - \varepsilon_n)d\alpha \qquad \ldots(4)$$

And
$$\overline{\int_a^b}(f_n + \varepsilon_n)d\alpha = \int_a^b (f_n + \varepsilon_n)d\alpha \qquad \ldots(5)$$

We know that,

$$\underline{\int_a^b} f d\alpha \quad \overline{\int_a^b} f d\alpha \qquad \ldots(6)$$

From (2),(3),(4),(5) and (6), we get that

$$\int_a^b (f_n - \varepsilon_n)d\alpha \quad \underline{\int_a^b} f d\alpha$$

$$\overline{\int_a^b} f d\alpha$$

$$\int_a^b (f_n + \varepsilon_n)d\alpha$$

Therefore,

$$\int_a^b f_n d\alpha - \varepsilon_n \int_a^b d\alpha \quad \int_a^b f d\alpha \quad \int_a^b f_n d\alpha + \varepsilon_n \int_a^b d\alpha$$

$$-\varepsilon_n[\alpha(b) - \alpha(a)] \quad \int_a^b f d\alpha - \int_a^b f_n d\alpha \quad \varepsilon_n[\alpha(b) - \alpha(a)]$$

Therefore ,

$$\left|\int_a^b f d\alpha - \int_a^b f_n d\alpha\right| \quad \varepsilon_n[\alpha(b) - \alpha(a)] > 0 \qquad \text{as } n> .$$

Therefore,

$$\lim_{n›} \int_a^b f_n d\alpha = \int_a^b f d\alpha ,$$

Hence ,

$$\lim_{n›} \int_a^b f_n d\alpha = \int_a^b (\lim_{n\to\infty} f_n)d\alpha .$$

Hence the theorem.

Theorem7.2

If $f_n \in R(\alpha)$ on [a, b], for n=1,2,3,.. and if

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad a \ x \ b,$$

and the series converges uniformly on [a, b], then

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n d\alpha .$$

In other words , the series can be integrated term by term.

Proof:

Let $s_n(x) = \sum_{k=1}^{n} f_k(x)$, n=1,2,3,…

Since $\sum f_n$ converges uniformly to f on[a, b],

$s_n›f$ uniformly on [a, b].

Therefore ,

by the above theorem7.1,

$$\int_a^b f d\alpha = \lim_{n›} \int_a^b s_n d\alpha$$

$$= \lim_{n›} \int_a^b \left(\sum_{k=1}^{n} f_k\right) d\alpha$$

$$= \lim_{n›} \left(\sum_{k=1}^{n} \int_a^b f_k d\alpha\right)$$

$$= \sum_{k=1}^{\infty} \int_a^b f_k d\alpha \; .$$

Hence the theorem.

7.3 Let us sum up

In this lesson, we have seen that

- If a sequence of integrable functions converges uniformly to a function, then the operations integration and limit can be interchanged.
- If a series of integrable functions converges uniformly to a limit function, then the series can be integrated term by term.

7.4 Lesson End Activities

1. Consider the sequence $\{f_n\}^{\propto}_{n-1}$ where $f_n(x) = nx(1-x^2)^n$ $0 \le x \le 1$. Prove that the limit of the integrals is not equal to the integral of the limit.

7.5 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

Lesson 8 Uniform convergence and Differentiation

8.0 Introduction
8.1 Aims and Objectives
8.2 Uniform convergence and Differentiation
8.3 Let us sum up
8.4 Lesson End Activities
8.5 References

8.0 Introduction

In this lesson, we are going to learn about the differentiability of the limit function of a sequence of functions.

8.1Aims and Objectives

After studying this lesson, you would know

- whether the limit function of a sequence of differentiable functions is differentiable
- If it is so, can the operations of differentiation and limit can be interchanged.

8.2 Uniform convergence and Differentiation

Uniform convergence of a sequence of differentiable functions $\{f_n\}$ implies nothing about the sequence $\{f_n'\}$.

We need stronger hypotheses for the assertion that:

f_n›f to imply f_n'› f '

i.e to interchange the operations of differentiation and limit.

Example:

Let $f_n(x)=(\sin nx)/ n$ x real,n=1,2,3,…

Then $f(x)=\lim_{n›}\ f_n(x)=0$ for all real x.

Then $f'(x)=0$

And $f_n'(x)=$ ncos nx

For x=0, $f_n'(0)=$ n › as n›

Hence $\{f_n'\}$ does not converge to f'.

Theorem8.1

Suppose $\{f_n\}$ is sequence of functions differentiable on [a, b] and such that $\{f_n(x_0)\}$ converges for some point x_0 in [a, b].

If $\{f_n'\}$ converges uniformly on [a, b],

then $\{f_n\}$ converges uniformly on [a, b], to a function f, and

$$f'(x)= \lim_{n›}\ f_n'(x), \qquad a \ \ x \ \ \ b.$$

Proof:

Since $\{f_n(x_0)\}$ converges for some x_0 in [a, b],

for given ε >0,

there exists an integer N_1 such that n N_1, m N_1 implies

$$|f_n(x_0) – f_m(x_0)|< ε/2 \qquad \text{…(1)}$$

Since $\{f_n'\}$ converge uniformly on [a, b],

$\{f_n'\}$ satisfies Cauchy criterion for uniform convergence.

Therefore for the same ε>0,

there exists an integer N_2 such that that n N_2, m N_2 implies

$$|f_n'(t) - f_m'(t)| < \frac{\varepsilon}{2(b-a)} \qquad a \quad t \quad b \qquad \ldots(2)$$

Let $N = \max(N_1, N_2)$
Let $x, \ t \in [a, b]$.
Applying mean value theorem to the function $(f_n - f_m)$, we get

$(f_n-f_m)(x) - (f_n-f_m)(t) = (f_n-f_m)'(\xi)\ (x-t)$, where ξ lies between x and t

Therefore,

$$|(f_n-f_m)(x) - (f_n-f_m)(t)| = |(f_n-f_m)'(\xi)\ (x-t)|$$

Therefore ,

for any x, t in [a, b], m N, n N implies

$$\begin{aligned} |f_n(x)-f_m(x)- f_n(t) + f_m(t)| &= |(f_n-f_m)'(\xi)|\,|(x-t)| \\ &\le \frac{\varepsilon}{2(b-a)}|x-t| \\ &\le \frac{\varepsilon}{2} \qquad \ldots(3) \end{aligned}$$

Therefore by choosing $t = x_0$,

for any x in [a, b] and m N, n N, we get

$$\begin{aligned} |f_n(x) - f_m(x)| &= |(f_n(x)-f_m(x)) - (f_n(x_0) - f_m(x_0)) + (f_n(x_0)-f_m(x_0))| \\ & \quad |(f_n(x)-f_m(x)) - (f_n(x_0) - f_m(x_0))| + |(f_n(x_0)-f_m(x_0))| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

Therefore $\{ f_n \}$ satisfies Cauchy criterion for uniform convergence.
Therefore $\{f_n\}$ converges uniformly on [a, b].

Let $\qquad f(x) = \lim_{n\to} f_n(x) \qquad a \quad x \quad b$
Fix a point x in [a, b].

Define $\varphi_n(t) = \dfrac{f_n(t) - f_n(x)}{t-x}, \qquad a \quad t \quad b,\ t \quad x \qquad \ldots(4)$

And $\quad \varphi(t) = \dfrac{f(t) - f(x)}{t-x}, \qquad a \quad t \quad b,\ t \quad x \qquad \ldots(5)$

Then,

$$\begin{aligned} \lim_{t\to x} \varphi_n(t) &= \lim_{t\to x} \frac{f_n(t) - f_n(x)}{t-x} \\ &= f_n'(x), \qquad n = 1,2,3,\ldots \qquad \ldots(6) \end{aligned}$$

and $\quad \lim_{t\to x} \varphi(t) = \lim_{t\to x} \dfrac{f(t) - f(x)}{t-x}$

$$= f'(x) \qquad \ldots(7)$$

From(3), for any x, t in [a, b], m N, n N implies

$$|(f_n(x) - f_n(t)) - (f_m(x) - f_m(t))| \le \frac{\varepsilon}{2(b-a)}|x-t|$$

Therefore,

$$|\varphi_n(t) - \varphi_m(t)| = \left|\frac{f_n(t) - f_n(x)}{t-x} - \frac{f_m(t) - f_m(x)}{t-x}\right|$$

$$\frac{\varepsilon}{2(b-a)}, \text{ for all t in [a, b], t } \quad x, \text{ m } \quad N, \text{ n } \quad N.$$

Hence $\{\varphi_n\}$ satisfies Cauchy criterion for uniform convergence, for t x.

Therefore $\{\varphi_n\}$ converges uniformly, for t x.

Also for t x,

$$\lim_{n\to} \varphi_n(t) = \lim_{n\to} \frac{f_n(t) - f_n(x)}{t - x}$$
$$= \frac{f(t) - f(x)}{t - x}$$
$$= \varphi(t).$$

Therefore ,

$$\lim_{t\to x} \varphi(t) = \lim_{t\to x} [\lim_{n\to} \varphi_n(t)]$$
$$= \lim_{n\to} [\lim_{t\to x} \varphi_n(t)] \qquad \text{[by theorem6.3.1]}$$
$$= \lim_{n\to} f_n'(x) \qquad \text{[from (6)]}$$

Therefore,

$$\lim_{t\to x} \varphi(t) = \lim_{n\to} f_n'(x) \qquad \ldots(8)$$

From (7) and (8), we get that

$$f'(x) = \lim_{t\to x} \varphi(t)$$
$$= \lim_{n\to} f_n'(x).$$

Hence the theorem.

Check your progress

Is there exist a real continuous function on the line which is nowhere differentiable.

Answer:

Yes.

Define $\varphi(x)=|x|$ (-1 x 1)

Extend φ to all real x by defining

$$\varphi(x+2)= \varphi(x) .$$

Then φ is continuous on R^1.

Define f on R^1 as

$$f(x)=\sum_{n=0}^{\infty}(\frac{3}{4})^n \varphi(4^n x)$$

f is the required function.

8.3Let us sum up

In this lesson, we have seen that

- just uniform convergence alone is not sufficient for the limit of a sequence of differentiable functions to be differentiable and
- the conditions required for interchanging limit and differentiation.

8.4 Lesson End Activities

1. Show that the sequence $\{f_n\}^{\infty}_{n-1}$ of functions where fn (x) = nx / $1+n^2x^2$, converges to f where f(x) = 0 $\forall$ x and that the equation $f^{`}(x)= lt_{n-\infty}\ fn^{`}(x)$ is true $\forall$ x $\neq 0$ but is false if x = 0.

8.5 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

LESSON - 9

EQUICONTINUOUS FAMILY OF FUNCTIONS

9.0 Introduction

In this lesson, we are going to study about pointwise bounded functions, uniformly bounded functions, equicontinuous family of functions.

9.1 Aims and Objectives

After studying this lesson, you should be able to identify

- Pointwise bounded functions
- Uniformly bounded functions
- Equicontinuous family of functions and
- Which of these families of functions contains a convergent subsequence.

9.2 Pointwise bounded functions

Definition

Let $\{f_n\}$ be a sequence of functions defined on a set E.
$\{f_n\}$ is said to be pointwise bounded on E, if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$.
In other words,
$\{f_n\}$ is said to be pointwise bounded on E, if there exists a finite valued function φ defined on E such that

$$|f_n(x)| < \varphi(x) \qquad x \in E, n=1,2,3,\dots.$$

Theorem 9.1

If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence $\{ f_{n_k} \}$ such that $\{ f_{n_k}(x) \}$ converges for every $x \in E$.

Proof

Let $\{f_n\}$ be a pointwise bounded sequence of complex functions on a countable set E.
Since E is countable,
The points of E can be arranged in a sequence as

$$\{x_i\}, \ i=1,2,3,\dots$$

Since $\{f_n\}$ is pontwise bounded on E,
$\{f_n(x_1)\}$ is a bounded sequence of complex numbers.
Therefore,
$\{f_n(x_1)\}$ contains a convergent subsequence.
Denote it by $\{f_{1,k}\}$i.e $\{f_{1,k}(x_1)\}$ converges as k›   .
Now consider $\{f_{1,k}(x_2)\}$.
Since $\{f_{1,k}\}$ is a subsequence of $\{f_n\}$,
$\{f_{1,k}\}$ is also pointwise bounded on E.

Therefore,

$\{f_{1,k}(x_2)\}$ contains a convergent subsequence, say $\{f_{2,k}(x_2)\}$.

Proceeding in this way, we get the sequences represented by the following array:

S_1: $f_{1,1}$ $f_{1,2}$ $f_{1,3}$ ……………
S_2: $f_{2,1}$ $f_{2,2}$ $f_{2,3}$ ……………
S_3: $f_{3,1}$ $f_{3,2}$ $f_{3,3}$ ……………
………………………………………
………………………………………
………………………………………

which have the following properties:

(1) S_n is a subsequence of S_{n-1}, for n=2,3,4,…
(2) $\{ \{f_{n,k}(x_n)\}$ converges as k›
(3) the order in which the functions appear is the same in each sequence. i.e if one function precedes another in S_1, then they are in same relation in every S_n, until one or the other is deleted.

Hence when going from one row in the above array to the next row below, functions may move left but never to the right.

Now consider the sequence

S: $f_{1,1}$ $f_{2,2}$ $f_{3,3}$ $f_{4,4}$ ……….

obtained by gong down the diagonal of the array.

By (3), the sequence S is a subsequence of S_n (except possibly its first n-1 terms), for n=1,2,3,…

Hence (2) implies that

$\{ f_{n,n}(x_i)\}$ converges for every $x_i \in E$.

Hence $\{f_n\}$ has a subsequence that converges for every $x_i \in E$.

Hence the theorem.

9.3 Uniformly bounded functions

Definition:

Let $\{f_n\}$ be a sequence of functions defined on a set E. $\{f_n\}$ is said to be uniformly bounded on E, if there exists a Number M such that

$$|f_n(x)| < M \qquad (x \in E, n=1,2,3,\ldots)$$

Check your progress

1. In section9.2, you have seen that pointwise bounded sequence of functions contains a convergent subsequence. Can you say that this will be true for uniformly bounded functions also?
2. Define $f_n(x) = \dfrac{x^2}{x^2+(1-nx)^2}$ (0 x 1, n=1,2,3,…) Show that $\{f_n\}$ is uniformly bounded on [0, 1].
3. Does every convergent sequence contain a uniformly convergent subsequence?

Answer

1. No.

Example

Let $f_n(x)$ = sin nx 0 x 2π, n=1,2,3,…

Then | $f_n(x)$ | = |sin nx| 1, 0 x 2π, n=1,2,3,…

Therefore,

$\{f_n\}$ is uniformly bounded on the compact set $[0,2\pi]$.

Suppose there exists a sequence $\{n_k\}$ such that

$\{\sin n_k x\}$ converges, for every $x \in [0,2\pi]$

Then $\lim_{k›} (\sin n_k x - \sin n_{k+1} x) = 0 \qquad 0 \quad x \quad 2\pi.$

Therefore,

$$\lim_{k›} (\sin n_k x - \sin n_{k+1} x)^2 = 0 \qquad 0 \quad x \quad 2\pi.$$

Therefore,

$$\lim_{k›} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1})^2 dx \qquad \ldots(1)$$

But if we evaluate $\int_0^{2\pi} (\sin n_k x - \sin n_{k+1})^2 dx$, we get

$$\int_0^{2\pi} (\sin n_k x - \sin n_{k+1})^2 dx$$

$$= \int_0^{2\pi} (\sin^2 n_k x + \sin^2 n_{k+1} - 2\sin n_k x \sin n_{k+1} x) dx$$

$$= 2\pi$$

Therefore,

$$\lim_{k›} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1})^2 dx = 2\pi, \text{ which contradicts (1).}$$

Therefore $\{f_n\}$ does not have any convergent subsequence.

2.

Example:

Let $f_n(x) = \dfrac{x^2}{x^2 + (1-nx)^2} \qquad (0 \quad x \quad 1,\ n=1,2,3,\ldots)$

Then $| f_n(x)| \quad 1.$

Hence $\{f_n\}$ is uniformly bounded on $[0, 1]$.

3. Need not be so.

Let $f_n(x) = \dfrac{x^2}{x^2 + (1-nx)^2} \qquad (0 \quad x \quad 1,\ n=1,2,3,\ldots)$

Then $\lim_{n›} f_n(x)=0 \qquad 0 \quad x \quad 1$

i.e $\{f_n\}$ converges pointwise to $f(x)=0$ on $[0, 1]$.

But $\{f_n\}$ does not contain any uniformly convergent subsequence, because

$f_n(1/n) = 1$ for all $n=1,2,3,\ldots.$

9.4 Equicontinuous family of functions

Definition

A family F of complex functions f defined on a set E in a metric space X is said to be equicontinuous on E if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$| f(x) - f(y) | < \varepsilon, \qquad \text{whenever } d(x,y) < \delta,\ x \in E,\ y \in E.$$

Here d denotes the metric of X.

Check your progress

1. Do you remember uniformly continuous function? Every member of an equicontinuous family of functions is uniformly continuous. True or False?

2. Let $f_n(x) = \dfrac{x^2}{x^2 + (1 - nx)^2}$ (0 x 1, n=1,2,3,…)

Is $\{f_n\}$ is a equicontinuous family of functions?

Answer

1. True.
2. No.

Theorem9.2

If K is a compact metric space, if $f_n \in C$ (K) for n=1,2,3,…and if $\{ f_n\}$ converges uniformly on K, then $\{ f_n\}$ is equicontinuous on K.

(Remark:

C(K) denotes the set of all continuous functions on K.)

Proof:

Let ε>0 be given.

Since $\{ f_n\}$ converges uniformly on K,

$\{ f_n\}$ satisfies the cauchy's condition for uniform convergence.

Hence for the given ε>0,

there exists an integer N such that m N, n N, $x \in K$ implies

$| f_n(x)- f_m(x)| <\varepsilon/3$.

Hence,

$| f_n(x)- f_N(x)| <\varepsilon/3$, whenever n>N and $x \in K$ ….(1)

Hence,

$\sup\{| f_n(x)- f_N(x)|,\ x \in K\} <\varepsilon/3$, whenever n>N.

Hence $\| f_n - f_N \| <\varepsilon/3$, (n>N) …(2)

Since continuous functions on a compact metric space are uniformly continuous and K is compact,

each f_i is a uniformly continuous function on K.

In particular $f_1,f_2,f_3,\dots,f_N$ are uniformly continuous on K.

Therefore for the same ε>0,

there exists a δ>0 such that

$| f_i(x) – f_i(y) | < \varepsilon/3$ whenever $d(x,y)< \delta$ and i=1,2,…N. …(3)

From (1) and (3), we get that,

if n>N and d(x, y)< δ,

$|f_n(x)-f_n(y)| =|f_n(x) – f_N(x) + f_N(x) – f_N(y) + f_N(y) – f_n(y) |$

$|f_n(x) – f_N(x)| + |f_N(x) – f_N(y) |+| f_N(y) – f_n(y) |$

$<\varepsilon/3+\varepsilon/3+\varepsilon/3 = \varepsilon$ ….(4)

From (2) and (4),

$|f_n(x)-f_n(y)|< \varepsilon$ whenever $d(x, y)< \delta$, $x \in K, y \in K$, n=1,2,3,….

Hence $\{ f_n\}$ is equicontinuous on K.

Hence the theorem.

Theorem9.3

If K is compact, if $f_n \in C(K)$ for n=1,2,3,…and if $\{ f_n\}$ is pointwise bounded and equicontinuous on K, then

(a) $\{ f_n\}$ is uniformly bounded on K

(b) $\{ f_n\}$ contains a uniformly convergent subsequence.

Proof:

Let K be compact and $f_n \in C(K)$, n=1,2,,3,..

Let { f_n} is pointwise bounded and equicontinuous on K.

(a) To prove that { f_n} is uniformly bounded on K.

Since { f_n} is equicontinuous on K,

for given ε> 0, there exists a δ>0 such that

$$|f_n(x)-f_n(y)|< \varepsilon, \quad \text{whenever } d(x, y)< \delta,\ n=1,2,3,\ldots \qquad \ldots.(1)$$

Since K is compact, for this δ>0, there exists a finite subset $\{p_1,p_2,\ldots p_r\}$ of K which is δ-dense in K.

i.e for every $x \in K$, there corresponds atleast one p_i with

$$d(x, p_i)< \delta \qquad \ldots.(2)$$

Since { f_n} is pointwise bounded on K,

{ $f_n(p_i)$} is bounded for all p_i.

i.e there exist M_i such that

$$| f_n(p_i)| <M_i \qquad i=1,2,\ldots r.$$

Let $M=Max(M_1,M_2,\ldots.M_r)$

Let x be any element of K.

Then by (2), there exists one p_i such that

$$d(x,p_i)< \delta$$

Then by (1),

$$|f_n(x)-f_n(p_i)|< \varepsilon, \qquad \text{for all n.}$$

Hence
$$| f_n(x)|=| f_n(x) - f_n(p_i) + f_n(p_i)|$$
$$| f_n(x) - f_n(p_i)| +| f_n(p_i)|$$
$$<\varepsilon+M_i, \qquad \text{for all n.}$$

Therefore { f_n} is uniformly bounded on K.

(b) To prove that { f_n} contains a uniformly convergent subsequence.

Since K is compact, K contains a countable dense subset, say E.

Then by the theorem9.1which states that,

"If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence { f_{n_k} } such that { $f_{n_k}(x)$ } converges for every $x \in E$.",

{ f_n} has a subsequence { f_{n_k} } such that { $f_{n_k}(x)$ } converges for every $x \in E$.

Let $f_{n_k} = g_i$, to simplify the notation.

Then { $g_i(x)$ } converges uniformly for every $x \in K$.

To prove that $\{g_i\}$ converges uniformly on K.

By (2), for every $x \in K$,

there corresponds atleast one p_i with

$$d(x, p_i) < \delta.$$

Let $V(x, \delta) = \{y \in K, d(x, y)< \delta\}$

Since E is dense in K and K is compact,

there are finitely many points $x_1,x_2,\ldots.x_m$ in E such that

$$K \subset V(x_1, \delta) \cup V(x_2, \delta) \cup \ldots\ldots \cup V(x_m, \delta)$$

Since { $g_i(x)$ } converges uniformly for every $x \in E$,

{ $g_i(x_s)$ } converges for s=1,2,…m.

Hence for each s=1,2,…m,

{ $g_i(x_s)$ } is a Cauchy sequence.

Hence for each s=1,2,…m, and ε>0,

there exists N_s such that

$$| g_i(x_s) - g_j(x_s) | < \varepsilon, \text{ whenever } i \ N_s , j \ \ N_s.$$

Let N=Max $(N_1, N_2, \ldots .N_m)$

Then $| g_i(x_s) - g_j(x_s) | < \varepsilon$, whenever i N , j N and s=1,2,,..m …(3)

Let x be any point in K.

Then $x \in V(x_s, \delta)$ for some s.

Hence $d(x,x_s) < \delta$.

Therefore by (2),

$$| g_i(x) - g_j(x_s) | < \varepsilon, \quad \text{for every I} \qquad \ldots(4)$$

From (3) and (4), for i N, j N, $x \in K$

$$| g_i(x) - g_j(x) | = | g_i(x) - g_i(x_s) + g_i(x_s) - g_j(x_s) + g_j(x_s) - g_j(x_s) |$$

$$| g_i(x) - g_i(x_s) |+ | g_i(x_s) - g_j(x_s) | +| g_j(x_s) - g_j(x_s) |$$

$$<\varepsilon+\varepsilon+\varepsilon = 3\varepsilon.$$

Therefore,

$\{g_i\}$ is uniformly convergent.

i.e $\{ f_{n_k} \}$ is uniformly convergent on K.

Hence $\{ f_n\}$ contains a uniformly convergent subsequence.

Hence the theorem.

9.5 Let us sum up

In this lesson we have studied about

- Pointwise bounded functions
- Uniformly bounded functions
- equicontinuous family of functions
- pointwise bounded sequence of functions on a countable set contains a convergent subsequence.
- Sequence of continuous functions on a compact metric space is equicontinuous.
- A sequence of pointwise bounded and equicontinuous functions on a compact space is uniformly bounded and contains a uniformly convergent subsequence.

9.6 Lesson End Activities

1. Prove that every uniformly convergent sequence of bounded functions is uniformly bounded.
2. Set $fn(x) = x^2/ x^2+ (1-nx)^2$; $0 \leq x \leq 1$, n=1,2,3… verify if the sequence of functions $\{f_n\}^{\propto}_{n-1}$ is equi continous.

9.7 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

Lesson 10 The Stone- Weierstrass Theorem

10.0 Introduction
10.1 Aims and Objectives
10.2 The Weierstrass Theorem
10.3 Algebra of functions
10.4 The Stone – Weierstrass theorem
10.5 Let us sum up.
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10.0 Introduction

In this lesson, we are going to study about the original version of the Weierstrass theorem and the Stone’s generalization of the Weierstrass Theorem.

10.1 Aims and Objectives

After studying this lesson, you would know

- Weierstrass Theorem
- Algebra of functions
- Uniform closure of an algebra
- Separation of points and
- Stone-Weierstrass Theorem.

10.2The Weierstrass Theorem

Theorem10.1

If f is a continuous complex function on [a, b], there exists a sequence of polynomials P_n such that

$$\lim_{n›} P_n(x) = f(x) \text{ uniformly on } [a, b].$$

If f is real, then P_n may be taken real.

Proof

Without loss of generality, assume that [a, b]=[0, 1].
Also assume that f(0) = f(1) = 0.
Suppose if f(0) 0, f(1) 0,

Consider $g(x)=f(x)-f(0) -x[f(1)-f(0)]$ 0 x 1

Then g(0)=g(1)=0.
Hence if g can be obtained as the limit of polynomials, then the same is true for f also.
Hence we will get the theorem.
Hence it is sufficient to prove the theorem with f(0) = f(1) = 0.

Let f be a continuous complex function on [0, 1] with f(0) =f(1) =0.
Since a continuous function on a compact space is uniformly continuous,
f is uniformly continuous on [0, 1].
Define f(x) to be zero for x outside [0, 1].
Then f is uniformly continuous on the whole real line.
Define $Q_n(x) = c_n(1-x^2)^n$, n=1,2,3,… …(1)
where c_n is chosen such that

$$\int_{-1}^{1} Q_n(x)dx = 1, \quad n=1,2,3,\dots \qquad \dots(2)$$

i.e

$$\int_{-1}^{1} c_n(1-x^2)^n dx = 1, \quad n=1,2,3,\dots \qquad \dots(3)$$

consider $\int_{-1}^{1}(1-x^2)^n\,dx = 2\int_{0}^{1}(1-x^2)^n\,dx$

$$2\int_{0}^{1/\sqrt{n}}(1-x^2)^n\,dx$$

$$2\int_{0}^{1/\sqrt{n}}(1-nx^2)dx$$

[since $(1-x^2)^n=1-nx^2+n(n-1)x^4/2-..$
$1-nx^2$]

$$=2[x-n\frac{x^3}{3}]_0^{1/\sqrt{n}}$$

$$=\frac{4}{3\sqrt{n}}$$

$$>\frac{1}{\sqrt{n}} \qquad \text{....(4)}$$

From (3) and (4),

$$1=\int_{-1}^{1}c_n(1-x^2)^n\,dx>\frac{c_n}{\sqrt{n}}$$

Therefore $c_n < \sqrt{n}$ ....(5)

Therefore for any δ>0,

$Q_n(x)=c_n(1-x^2)^n$

$\sqrt{n}\,(1-x^2)^n$

$\sqrt{n}\,(1-\delta^2)^n$ for $\delta \quad |x| \quad 1.$ ....(6)

Therefore,

$Q_n›\ 0$ uniformly in $\delta \quad |x| \quad 1$

Define $P_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)dt$

Since f is zero outside [0, 1],

$$P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)dt$$

Changing the variable t as t-x, we get

$$P_n(x) = \int_{0}^{1} f(t)Q_n(t-x)dt\,, \qquad \text{....(7)}$$

which is a polynomial in x.

Thus $\{P_n\}$ is a sequence of polynomials, which are real if, f is real.

Since f is uniformly continuous,

for given ε > 0, there exists a δ > 0 such that

$|y-x|<\delta$ implies $|f(y) - f(x)| < \varepsilon/2$. ....(8)

Let M=sup |f(x)|

For 0 x 1,

$$|P_n(x) - f(x)| = \left|\int_{-1}^{1} f(x+t)Q_n(t)dt - f(x)\right|$$

$$=\left|\int_{-1}^{1} f(x+t)Q_n(t)dt - \int_{-1}^{1} f(x)Q_n(t)dt\right|$$

$$=\left|\int_{-1}^{1}[f(x+t)-f(x)]Q_n(t)dt\right|$$

$$\int_{-1}^{1}|f(x+t)-f(x)|\,Q_n(t)dt \qquad [\text{since } Q_n(t)\ \ 0\ ]$$

$$= \int_{-1}^{-\delta} | f(x+t) - f(x) | Q_n(t)dt + \int_{-\delta}^{\delta} | f(x+t) - f(x) | Q_n(t)dt +$$
$$\int_{\delta}^{1} | f(x+t) - f(x) | Q_n(t)dt$$

Now $|f(x+t) - f(x)|$ $|f(x+t)|+|f(t)|$
$2M$

Hence ,

$$|P_n(x) - f(x)| \quad 2M\int_{-1}^{-\delta} Q_n(t)dt + \frac{\varepsilon}{2}\int_{-\delta}^{\delta} Q_n(t)dt + 2M\int_{\delta}^{1} Q_n(t)dt$$

$$2M\sqrt{n}\,(1-\delta^2)^n \int_{-1}^{-\delta} dt + \frac{\varepsilon}{2} + 2M\sqrt{n}\,(1-\delta^2)^n \int_{\delta}^{1} dt$$

$$2M\sqrt{n}\,(1-\delta^2)^n(-\delta+1) + \frac{\varepsilon}{2} + 2M\sqrt{n}\,(1-\delta^2)^n(1-\delta\}$$

$$4M\sqrt{n}\,(1-\delta^2)^n + \frac{\varepsilon}{2}$$

$<\varepsilon$ for sufficiently large n.

[since $\lim_{n›}$ $(1-\delta^2)=0$]

Hence,

$$\lim_{n›} \quad P_n(x)=f(x)$$

Hence the theorem.

Theorem10.2

For every interval [-a, a] there is a sequence of real polynomials P_n such that

$\lim_{n›}$ $P_n(x) = |x|$ uniformly on [-a, a].

Proof

Since $|x|$ is a continuous function on [-a, a], by the above theorem 10.1, there exist a sequence $\{P_n^*\}$ of real polynomials which converges to $|x|$ uniformly on [-a, a].

i.e $\lim_{n›}$ $P_n^*(x) =|x|$ uniformly on [-a, a]. …(1)

Therefore ,

$\lim_{n›}$ $P_n^*(0) =0$ …(2)

Let $P_n(x)= P_n^*(x)- P_n^*(0)$

Then from (1) and (2),

$$P_n(0)= P_n^*(0) - P_n^*(0) = 0.$$

and $\lim_{n›}$ $P_n(x) = \lim_{n›}$ $[P_n^*(x) - P_n^*(0)]$
$=|x| -0$
$=|x|$ uniformly on [-a, a].

Hence the theorem.

10.3 Algebra of functions

Definition

A family A of complex functions defined on a set E is said to be an algebra if

(i) $f+g \in A$

(ii) $fg \in A$ and

(iii) $cf \in A$

for all $f\in A$, $g\in A$ and for all complex constants c.

i.e if A is closed under addition, multiplication and scalar multiplication.

Remark

If we consider algebras of real functions,

then (iii) is required to hold for all real c.

Definition

If $\mathcal{A}$ has the property that $f \in \mathcal{A}$ whenever

(i) $f_n \in \mathcal{A}$ (n=1,2,3,..) and

(ii) $f_n \to f$ uniformly on E,

then $\mathcal{A}$ is said to be uniformly closed.

Definition

Let $\mathcal{B}$ be the set of all functions which are limits of uniformly convergent sequences of members of $\mathcal{A}$. Then $\mathcal{B}$ is called uniform closure of $\mathcal{A}$.

Thorem10.3

Let $\mathcal{B}$ be the uniform closure of an algebra $\mathcal{A}$ of bounded functions.

Then $\mathcal{B}$ is a uniformly closed algebra.

Proof

Let $f \in \mathcal{B}$ and $g \in \mathcal{B}$.

Then there exists uniformly convergent sequences of functions $\{f_n\}$ and $\{g_n\}$ such that

$f_n \to f$ uniformly and $g_n \to g$ uniformly.

Hence $f_n+g_n \to f+g$ uniformly,

$f_n g_n \to fg$ uniformly

and $cf_n \to cf$ uniformly, where c is a constant.

Hence

$f+g \in \mathcal{A}$,

$fg \in \mathcal{A}$ and

$cf \in \mathcal{A}$.

Hence $\mathcal{B}$ is uniformly closed.

Definition

Let $\mathcal{A}$ be a family of functions on a set E.

Then $\mathcal{A}$ is said to separate points on E if, to every pair of distinct points $x_1, x_2 \in E$, there corresponds a function $f \in \mathcal{A}$ such that

$$f(x_1) \quad f(x_2).$$

Definition

Let $\mathcal{A}$ be a family of functions on a set E. $\mathcal{A}$ is said to vanish at no point of E if, to each $x \in E$, there corresponds a function $g \in \mathcal{A}$ such that $g(x) \quad 0$.

Check your progress

1.Can you give an example of an algebra of functions which separates points and vanishes at no point .

2.Can you give an example of algebra of functions which does not separate points.

Answer

1.The algebra of polynomials in one variable has these properties on R^1.

2.The set of all even polynomials on [-1, 1].

Theorem 10.4

Suppose A is an algebra of functions on a set E , A separates points on E, and A vanishes at no point of E.

Suppose x_1, x_2 are distinct points of E and c_1, c_2 are constants (real if A is a real algebra).
Then A contains a function f such that

$$f(x_1) = c_1,\ f(x_2) = c_2$$

Proof

Let x_1, x_2 be distinct points of E.
Since A separates points on E,
A Contains functions g, h, k such that

$$g(x_1)\quad g(x_2),$$
$$h(x_1)\quad 0,$$
$$k(x_2)\quad 0. \qquad \text{….(1)}$$

Let $u = gk - g(x_1)k$ and
$V = gh - g(x_2)h$.

Since A is an algebra,
g,h,k ∈ A implies u ∈ A , v ∈ A .

Also $u(x_1) = g(x_1)k(x_1) - g(x_1)k(x_1) = 0$,
$v(x_2) = g(x_2)h(x_2) - g(x_2)h(x_2) = 0$,
$u(x_2) = g(x_2)k(x_2) - g(x_1)k(x_2)$
$= [g(x_2) - g(x_1)]k(x_2)$
0 [from (1)]
$v(x_1) = g(x_1)h(x_1) - g(x_2)h(x_1)$
$= [g(x_2) - g(x_1)]h(x_1)$
0 [from (1)]

Define f on E as

$$f = \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}.$$

Since u, v ∈ A and A is an algebra,
f ∈ A .

And $$f(x_1) = \frac{c_1 v(x_1)}{v(x_1)} + \frac{c_2 u(x_1)}{u(x_2)}$$
$= c_1 + 0$
$= c_1,$

$$f(x_2) = \frac{c_1 v(x_2)}{v(x_1)} + \frac{c_2 u(x_2)}{u(x_2)}$$
$= 0 + c_2$
$= c_2.$

Hence f has the desired properties.
Hence the theorem.

10.4 The Stone – Weierstrass theorem

Theorem 10.5 Stone's generalization of the theorem.

Let $\mathcal{A}$ be an algebra of real continuous functions on a compact set K. If $\mathcal{A}$ separates points on K and if $\mathcal{A}$ vanishes at no point of K, then the uniform closure $\mathcal{B}$ of $\mathcal{A}$ consists of all real continuous functions on K.

Proof

We shall divide the proof into four steps.

Step1: If $f \in \mathcal{B}$, then $|f| \in \mathcal{B}$.

Proof:

Let $a=\sup\{|f(x)|, x \in K\}$ …(1)

Let ε>0 be given.

Then by the therem 10.2 , which states that,

"For every interval [-a, a] there is a sequence of real polynomials P_n such that $\lim_{n›} P_n(x) = |x|$ uniformly on [-a, a]. ",

there exists a sequence of real polynomials $\{P_n\}$ such that

$P_n(0)=0$ and $\lim_{n›} P_n(y)=|y|$ uniformly on [-a, a].

Therefore,

there exists real numbers c_1, c_2, …c_n such that

$$\left|\sum_{i=1}^{n} c_i y^i - |y|\,\right| < \varepsilon \qquad -a \quad y \quad a \qquad \ldots(2)$$

Since $\mathcal{B}$ is an algebra,

the function $g = \sum_{i=1}^{n} c_i f^i$ is a member of $\mathcal{B}$.

Therefore,

$$|\,g(x) - |f(x)|\,| = \left|\sum_{i=1}^{n} c_i f^i(x) - |f(x)|\,\right|$$

$$= \left|\sum_{i=1}^{n} c_i y^i - |y|\,\right|, \quad \text{where } y=f(x). \qquad \ldots(3)$$

Since $a = \sup\{ |f(x)|, x \in K \}$

$-a \quad f(x) \quad a$, for all $x \in K$.

i.e $-a \quad y \quad a$.

Hence by(2),

$$\left|\sum_{i=1}^{n} c_i y^i - |y|\,\right| < \varepsilon \qquad \ldots(4)$$

From (3) and (4),

$|\,g(x) - |f(x)|\,| < \varepsilon$ for all $x \in K$.

This implies that

$|f| \in \mathcal{B}$, since $\mathcal{B}$ is uniformly closed.

Hence the claim that,

if $f \in \mathcal{B}$, then $|f| \in \mathcal{B}$.

Step 2: If f $\in \mathcal{B}$ and g $\in \mathcal{B}$, then max(f,g) $\in \mathcal{B}$ and min(f,g) $\in \mathcal{B}$.

Proof:

By definition,

$$\max(f, g)(x) = \begin{cases} f(x) & \text{if } f(x)\ g(x) \\ g(x) & \text{if } f(x)<g(x) \end{cases}$$

Therefore ,

$$\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2}$$

Similarly $\min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2}$

Since f $\in \mathcal{B}$, g $\in \mathcal{B}$, and $\mathcal{B}$ is closed ,

f + g $\in \mathcal{B}$

and f - g $\in \mathcal{B}$.

By the proof of step 1,

| f – g | $\in \mathcal{B}$.

Hence,

$$\max(f, g) = \frac{f+g}{2} + \frac{|f-g|}{2} \in \mathcal{B}$$

and $$\min(f, g) = \frac{f+g}{2} - \frac{|f-g|}{2} \in \mathcal{B}\ .$$

Hence step 2 is proved.

By repeating this, we get that if $f_1, f_2, \ldots.. f_n \in \mathcal{B}$,

then $\max(f_1, f_2, \ldots.. f_n) \in \mathcal{B}$

and $\min(f_1, f_2, \ldots.. f_n) \in \mathcal{B}$

Step 3: Given a real valued function f, continuous on K, a point x $\in$ K and ε > 0, there exists a function $g_x \in \mathcal{B}$ such that

$g_x(x) = f(x)$

and $g_x(t) > f(t) - \varepsilon$ for all t $\in$ K. …(5)

Proof:

Let x be the given point in K.

Then for every y in K with y x, by theorem10.4, taking c_1=f(x) and c_2=f(y), we get that, there exists a function $h_y \in \mathcal{A}$ satisfying

$h_y(x) =$ f(x) and

$h_y(y) =$ f(y). …(6)

Since $\mathcal{A} \subset \mathcal{B}$, $h_y \in \mathcal{B}$.

Since $h_y$is continuous, there exists an open set J_y containing y such that

$h_y(t) > f(t) - \varepsilon$ for all t $\in J_y$ …(7)

Then $K \subset \bigcup_{y\in K} J_y$.

Since K is compact,

there is a finite set of points $y_1, y_2, \ldots y_n$ such that

$K \subset J_{y_1} \cup J_{y_2} \cup \ldots. \cup J_{y_n}$.

Let $g_x = \max(h_{y_1}, h_{y_2} .....h_{y_n})$

By step 2, $g_x \in \mathscr{B}$.

Now from(6),

$$h_{y_i}(x) = f(x) \text{ for all } i=1,2,\dots n .$$

Therefore,

$$g_x(x) = \max(h_{y_1}, h_{y_2} .....h_{y_n})(x) = f(x).$$

Also from(7),

$$h_{y_i}(t) > f(t) - \varepsilon, \quad \text{for all } i=1,2,\dots n.$$

Therefore,

$$g_x(t) = \max(h_{y_1}, h_{y_2} .....h_{y_n})(t)$$
$$> f(t) - \varepsilon, \quad \text{for all } t \in K.$$

Hence g_x has the desired properties.

Hence step 3 is proved.

Step 4:

Given a real valued function f continuous on K and $\varepsilon > 0$, there exists a function $h \in \mathscr{B}$ such that

$$|h(x) - f(x)| < \varepsilon \quad \text{for all } x \in K.$$

i.e the uniform closure $\mathscr{B}$ of $\mathscr{A}$ consists of all real continuous functions on K.

Proof:

Consider the functions g_x, for each $x \in K$ constructed in step 3.

Since each g_x is continuous on K,

there exist open sets V_x containing x such that

$$g_x(t) < f(t) + \varepsilon \quad \text{for all } t \in V_x \qquad \dots(8)$$

Then $K \subset \bigcup_{x \in K} V_x$.

Since K is compact,

there exists a finite set of points $x_1, x_2, .. x_m$ such that

$$K \subset V_{x_1} \cup V_{x_2} \cup ..... \cup V_{x_m}$$

Let $h = \min(g_{x_1}, g_{x_2}, ...g_{x_m})$

Then $h \in \mathscr{B}$, by step 2.

By (5), $g_{x_i}(t) > f(t) - \varepsilon$, for all $t \in K$, $i=1,2,..n$.

Therefore,

$$h(t) > f(t) - \varepsilon, \text{ for all } t \in K. \qquad \dots(9)$$

By (8), $g_{x_i}(t) < f(t) + \varepsilon$, for all $t \in K$, $i=1,2,..n$

Therefore,

$$h(t) < f(t) + \varepsilon, \text{ for all } t \in K. \qquad \dots(10)$$

From(9) and (10),

$$f(t) - \varepsilon < h(t) < f(t) + \varepsilon, \quad \text{for all } t \in K.$$

Hence ,

$$|h(t) - f(t)| < \varepsilon, \quad \text{for all } t \in K.$$

Hence for every real function f continuous on K and $\varepsilon>0$, there exists a function $h \in \mathscr{B}$ such that

$$|h(t) - f(t)| < \varepsilon, \quad \text{for all } t \in K.$$

Hence the uniform closure B of A consists of all real continuous functions on K.
Hence the theorem.

10.5 Let us sum up

In this lesson, we have seen

- The original version of Weierstrass Theorem which says that every continuous function (real or complex) on a finite interval is the limit of a sequence of polynomials.
- Definitions of algebra of functions and Uniform closure of an algebra
- Definition of Separation of points and Separation theorem and
- The Stone-Weierstrass Theorem and its proof.

10.6 Lesson End Activities

1. if f is continous on [0, 1] and if $_0\int^1 f(x)\, x^n\, dx = 0$ (n=0,1,2,..). Prove that f(x) =0 on [0,1].
2. let K be the unit circle in the complex plane and set A be the algebra of all function of the form.

   $F(e^{io}) = \sum^{N} Cn\, e^{in\theta}$ (θ real) then prove tht A sepraties points on K and that A vanishes at no point of K but there are continous functions on K which are not in the uniform closure of A.

10.7 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

UNIT III

FUNCTIONS OF SEVERAL VARIABLES

LESSON – 11

LINEAR TRANSFORMATIONS

Contents

11.0 Introduction

In this lesson, first we are going to be introduced to the definition of a vector space and some of its properties , which you might have studied in UG classes and then w are going to see about linear transformations, linear operators and the matrix representation of a linear transformation.

11.1 Aims and objectives

After studying this lesson you would know

- What is a linear transformation?
- What is a linear operator?
- Inverse of an operator.
- Some of the properties of Vector spaces, linear transformations and inverse operators and
- How to represent a linear transformation in the matrix form?

11.2 Vector Spaces

Definition

A nonempty set $X \subset R^n$ is asid to be a vector space if,

(i) $x + y \in X$ and
(ii) $cx \in X$ for all $x \in X$,$y \in X$ and for all scalars c.

Definition

If $x_1, x_2, \ldots x_k \in R^n$ and $c_1, c_2, \ldots . c_k$ are scalars ,
the vector $c_1x_1 + c_2x_2 + \ldots + c_nx_k$ is called a linear combination of $x_1, x_2, \ldots x_k$.

Definition

If $S \subset R^n$ and if E is the set of all linear combinations of elements of S, then we say that S spans E or that E is the span of S.

Definition

A set consisting of vectors $x_1, x_2, \ldots x_k$ is said to be independent if the relation
$c_1x_1 + c_2x_2 + \ldots + c_nx_k = 0$ implies that $c_1 = c_2 = \ldots . = c_k = 0$
Otherwise $\{ x_1, x_2, \ldots x_k \}$ is said to be linearly dependent.

Definition

If a vector space X contains an independent set of r vectors but contains no independent set of r+1 vectors, then we say that X has dimension r .

We write this as dim X = r.

Definition

An independent subset of a vector space X which spans X is called a basis of X.

Remarks:

1.The set consisting of 0 alone is a vector space and its dimension is 0.

2.If $B = \{ x_1, x_2, \ldots x_r \}$ is a basis of X,
then B spans X and B is independent.

Hence every $x \in X$ has a unique representation of the form

$$x = c_1x_1 + c_2x_2 + \ldots + c_nx_r.$$

The numbers $c_1, c_2, \ldots . c_r$ are called the coordinates of X with respect to the basis B.

Example:

The set $\{ e_1, e_2, \ldots . e_n \}$,
where e_j is the vector in R^n whose jth coordinate is 1 and whose other coordinates are all 0.

i.e $e_1 = (1, 0, 0, \ldots . 0)$
$e_2 = (0, 1, 0, \ldots . 0)$
…………………
…………………
and $e_n = (0, 0, 0, \ldots . 1)$

Now, if $x = (x_1, x_2, \ldots x_n) \in R^n$,
then ,

$$x = x_1(1, 0, 0, \ldots . 0) + x_2(0, 1, 0, \ldots . 0) + \ldots . x_n(0, 0, 0, \ldots . 1)$$
$$= x_1e_1 + x_2e_2 + \ldots . x_ne_n.$$

Hence $\{ e_1, e_2, \ldots . e_n \}$ forms a basis of R^n.
The set $\{ e_1, e_2, \ldots . e_n \}$is called the standard basis of R^n.

Some important results in vector spaces:

1. Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then dim X r.
2. dim $R^n = n$.

3. Suppose X is a vector space , and dim X = n.
    (a) A set E of n vectors in X spans X if and only if E is independent.
    (b) X has a basis and every basis consists of n vectors.
    (c) If 1 r n and { y_1, y_2, …y_r} is an independent set in X, then X has a basis containing { y_1, y_2, …y_r}.

Check your progress

1.Every span of a set is a vector space. True or False?

2.dim R^3 = ?

11.3 Linear transformations and linear operators

Definition

A mapping A of a vetor space X into a vector space Y is said to be a linear transformation if

(i) $A(x_1 + x_2) = A(x_1) + A(x_2)$ and

(ii) $A(cx)=cA(x)$, for all x, x_1, x_2 $\in$ X and all scalars c.

*If A is Linear , then A(x) is simply written as Ax.

*If A is linear A0=0.

Remark:

A linear transformation A of X into Y is completely determined by its action on any basis.

Proof

If { $x_1, x_2, \ldots x_n$ } is s basis of X,

then every x $\in$ X has a unique representation of the form

$$x = \sum_{i=1}^{n} c_i x_i \ .$$

Since A is linear,

$$Ax = A\left(\sum_{i=1}^{n} c_i x_i\right)$$

$$= \sum_{i=1}^{n} c_i Ax_i \ .$$

Hence Ax can be computed from the vectors Ax_1, Ax_2,….Ax_n and the coordinates $c_1, c_2, \ldots . c_n$.

i.e R(A) is the span of the set Q= { Ax_1, Ax_2,….Ax_n }

Definition

Linear transformations of X into X are called linear operators on X.

Definition

If A is a linear operator on X such that

(i) A is one-to-one and

(ii) A maps X onto X,

then A is said to be invertible and the operator A^{-1} on X is defined such that

$A^{-1}(Ax) = x$ for all x $\in$ X.

Check your progress

1.Prove that $A(A^{-1}x)= x$ for all x $\in$X.

2.prove that A^{-1} is linear

Theorem 11.1

A linear operator A on a finite dimensional vector space X is one–to-one if and only if the range of A is all X.

Proof

Let X be a vector space of dimension n and

$\{ x_1, x_2, \ldots x_n \}$ be a basis of X.

Since A is linear,

$\mathcal{R}(A)$ is the span of the set Q=$\{ Ax_1, Ax_2, \ldots. Ax_n \}$.

We know that

"A set E of n vectors in X spans X if and only if E is independent".

Hence $\mathcal{R}(A) = X$ if and only if Q is independent.

Hence, it is enough to prove that Q is independent if and only if A is one-to-one.

To prove that Q is independent if and only if A is one-to-one .

Assume that A is one-to-one.

Suppose $\sum_{i=1}^{n} c_i A x_i = 0$,

Then ,

$$A\left(\sum_{i=1}^{n} c_i x_i\right) = \sum_{i=1}^{n} c_i A x_i = 0. \text{ [since A is linear]}$$

Since A is one-to-one, this implies that

$$\sum_{i=1}^{n} c_i x_i = 0$$

Since $\{ x_1, x_2, \ldots x_n \}$ is linearly independent, this implies that

$$c_1 = c_2 = \ldots.. = c_n = 0.$$

Hence Q=$\{ Ax_1, Ax_2, \ldots. Ax_n \}$ is linearly independent.

Conversely, assume that Q is linearly independent.

Suppose if ,

$$A\left(\sum_{i=1}^{n} c_i x_i\right) = 0.$$

Then $\sum_{i=1}^{n} c_i A x_i = 0$ [since A is linear]

This implies that $c_1 = c_2 = \ldots.. = c_n = 0$. [since Q is independent]

This implies that $x = \sum_{i=1}^{n} c_i x_i = 0$

i.e Ax=0 only if x=0 …(1)

Therefore,

if Ax = Ay,

Then A(x-y)=Ax – Ay =0

Then by (1), x-y=0.

Hence x=y.

Hence A is one-to-one.

Hence the result

Hence A is one-to-one if and only if $\mathcal{R}(A) = X$.

Hence the theorem.

Definition

$L(X, Y)$ denote the set of all linear transformations of the vector space X into the vector space Y.

$L(X, X)$ can be simply written as $L(X)$.

i.e $L(X)$ denotes the set of all operators on X.

Definition

If $A_1, A_2 \in L(X, Y)$ and if c_1, c_2 are scalars,

$c_1A_1 + c_2A_2$ is defined by

$$(c_1A_1 + c_2A_2)(x) = c_1A_1x + c_2A_2x, \quad \text{for } x \in X.$$

Since A_1 and A_2 are linear, $c_1A_1 + c_2A_2$ is also linear.

Hence $c_1A_1 + c_2A_2 \in L(X, Y)$

Definition

If X, Y, Z are vector spaces and

if $A \in L(X,Y)$, $B \in L(Y,Z)$,

Their product BA is defined to be the composition of A and B.

i.e $(BA)(x) = B(Ax)$, for all $x \in X$.

Then $BA \in L(X, Y)$

Definition

For $A \in L(R^n, R^m)$, the norm of A is defined as

$$||A|| = \sup \{ |A(x)|, x \in R^n \text{ with } |x| \quad 1 \}$$

Remark:

$$|A(x)| \quad ||A||\,|x|, \qquad \text{for all } x \in R^n$$

Proof

Let $x \in R^n$.

Define $y = \frac{x}{|x|}$

Then $|y| = 1$.

Therefore by definition of $||A||$,

$$|A(y)| \quad ||A||$$

Therefore ,

$$|A(\frac{x}{|x|})| \quad ||A||$$

Therefore ,

$$\frac{|A(x)|}{|x|} \quad ||A||$$

Therefore ,

$$|A(x)| \quad ||A||\,|x|, \qquad \text{for all } x \in R^n.$$

Hence the result.

Also, if λ is such that $|A(x)| \quad \lambda |x|$, for all $x \in R^n$,

then $||A|| \quad \lambda$.

Check your progress

1.If $A_1, A_2 \in L(X, Y)$ and if c_1, c_2 are scalars,
prove that $c_1A_1 + c_2A_2$ is linear i.e $c_1A_1 + c_2A_2 \in L(X, Y)$
2.If $A \in L(X,Y)$, $B \in L(Y,Z)$, prove that $BA \in L(X, Y)$
3.If X= Y =Z, is BA = AB?

Answers

1.For $x_1, x_2 \in X$ and scalars a,b , consider

$$(c_1A_1 + c_2A_2)(ax_1+bx_2) = c_1A_1 (ax_1+bx_2)+c_2A_2(ax_1+bx_2)$$

[by definition of $c_1A_1 + c_2A_2$]

$$= c_1 (aA_1x_1+bA_1x_2)+ c_2 (aA_2x_1+bA_2x_2)$$
$$= c_1aA_1x_1+c_1bA_1x2+ c_2aA_2x_1+ c_2 bA_2x_2$$
$$=a(c_1A_1 +c_2A_2)(x_1) +b(c_1A_1 +c_2A_2)(x_2)$$

Hence $c_1A_1 + c_2A_2$ is linear.
2.Prove it by yourself.
3. BA need not be the same as AB.

Theorem11.2

(a)If $A \in L(R^n, R^m)$,
then $\|A\| < \infty$ (i.e $\|A\|$ is finite) and
A is a uniformly continuous mapping of R^n into R^m.
(b)If $A, B \in L(R^n, R^m)$ and c is a scalar,
then $\|A+B\| \le \|A\| + \|B\|$,
$\|cA\| = |c|\,\|A\|$.
If we define distance between A and B as $\|A - B\|$,
then $L(R^n, R^m)$ is a metric space.
(c) If $A \in L(R^n, R^m)$ and $B \in L(R^n, R^m)$,
then $\|BA\| \le \|B\|\,\|A\|$.

Proof

(a) Let $A \in L(R^n, R^m)$.
Let $\{e_1 , e_2 , \ldots e_n\}$ be the basis in R^n.

Suppose that $x = \sum_{i=1}^{n} c_i e_i$ with $|x| \le 1$.

Then $|c_i| \le 1$ for all i=1, 2, ...n.
Therefore ,

$$|A(x)| = |\, A(\sum_{i=1}^{n} c_i e_i)|$$
$$= |\sum_{i=1}^{n} c_i Ae_i \,|$$
$$\le \sum_{i=1}^{n} c_i \,|\, Ae_i \,|$$
$$\le \sum_{i=1}^{n} |\, Ae_i \,|$$

Therefore,

$$\sup \{ |A(x)|, x \in R^n \text{ with } |x| \le 1 \} \le \sum_{i=1}^{n} |\, Ae_i \,| < \infty .$$

i.e $\|A\| < \infty$.

To prove that A is uniformly continuous

From the inequality $|A(x)| \le \|A\|\,|x|$, for all $x \in R^n$,

$$|Ax - Ay| \le \|A\|\,|x - y|, \text{ for all } x, y \in R^n.$$

Therefore for any $\varepsilon > 0$,

if we choose $\delta = \dfrac{\varepsilon}{\|A\|}$ (independent of x and y), then

$|x - y| < \delta$ implies that $|Ax - Ay| \le \|A\|\,|x - y| < \varepsilon$.

Hence for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$|Ax - Ay| < \varepsilon$ whenever $|x - y| < \delta$, $x, y \in R^n$.

Hence A is uniformly continuous.

(b)To prove that, if $A, B \in L(R^n, R^m)$ and c is a scalar,

then $\|A+B\| \le \|A\| + \|B\|$ and $\|cA\| = |c|\,\|A\|$.

Consider $|(A+B)(x)| = |Ax+Bx|$

$\le |Ax| + |Bx|$

$\le \|A\|\,|x| + \|B\|\,|x|$

Therefore ,

if $|x| \le 1$,

Then $|(A+B)(x)| \le \|A\| + \|B\|$

Therefore,

$\sup\{\, |(A+B)(x)|,\ x \in R^n, |x| \le 1\} \le \|A\| + \|B\|$

Therefore ,

$$\|A+B\| \le \|A\| + \|B\|$$

Consider $|(cA)(x)| = |c(Ax)|$

$= |c|\,|Ax|$.

Therefore,

$\sup\{\, |(cA)(x)|, x \in R^n \text{ and } |x| \le 1\} = |c| \sup\{\, |Ax|, x \in R^n \text{ and } |x| \le 1\}$

Therefore,

$$\|cA\| = |c|\,\|A\|.$$

To prove that $L(R^n, R^m)$ is a metric space.

Define the metric on $L(R^n, R^m)$ as follows:

If $A, B \in L(R^n, R^m)$,

define the distance between A and B as $\|A - B\|$.

Then

1. $\|A-B\| = \sup\{|(A-B)(x)|,\ x \in R^n \text{ and } |x| \le 1\} \ge 0$

2. $\|A-B\| = 0$ if and only if $\sup\{|(A-B)(x)|,\ x \in R^n \text{ and } |x| \le 1\} = 0$

   if and only if $|(A-B)(x)| = 0$, for all $x \in R^n$ with $|x| \le 1$.

   if and only if $|Ax-Bx| = 0$, for all $x \in R^n$ with $|x| \le 1$

   if and only if $Ax=Bx$, for all $x \in R^n$ with $|x| \le 1$ ..(1)

Suppose if $y \in R^n$, with $y \ne 0$,

then $x = \dfrac{y}{|y|}$ is such that $|x|=1$

Therefore, $Ax = Bx$

implies $A\left(\dfrac{y}{|y|}\right) = B\left(\dfrac{y}{|y|}\right)$

implies $\frac{1}{|y|}Ay = \frac{1}{|y|}By$

implies $Ay = By$ [Since $|y|\ \ 0$, we can divide by $|y|$]

Using this in (1), we get that

$\|A-B\| = 0$ if and only if $Ay = By$, for all $y \in R^n$.

Hence $\|A-B\| = 0$ if and only if $A = B$.

3. $\|A-B\|$ = $\sup\{|(A-B)(x)|,\ x \in R^n \text{ and } |x|\ \ 1\}$
= $\sup\{|(B-A)(x)|,\ x \in R^n \text{ and } |x|\ \ 1\}$
= $\|B-A\|$

4. $\|A-B\|$ = $\sup\{|(A-B)(x)|,\ x \in R^n \text{ and } |x|\ \ 1\}$
= $\sup\{|(Ax-Bx|,\ x \in R^n \text{ and } |x|\ \ 1\}$
= $\sup\{|(Ax-Cx+Cx-Bx|,\ x \in R^n \text{ and } |x|\ \ 1\}$

$\sup\{|(Ax-Cx\ |,\ x \in R^n \text{ and } |x|\ \ 1\}$
$+ \sup\{|(Cx-Bx|,\ x \in R^n \text{ and } |x|\ \ 1\}$
$\|A-C\| + \|C-B\|$

Hence the triangle inequqlity is satisfied.

Hence the metric we defined on $L(R^n, R^m)$ satisfies all the required properties.
Hence $L(R^n, R^m)$ is a metric space.
Hence (b) is proved.

(c) To prove that, if $A \in L(R^n, R^m)$ and $B \in L(R^n, R^m)$,
then $\|BA\|\ \ \ \|B\|\ \|A\|$.

Consider $|(BA)(x)| = |B(Ax)|$
$|B\|\ |Ax|$
$\|B\|\ \|A\|\ |x|$

Therefore ,
$\sup\{|(BA)(x)|,\ x \in R^n \text{ and } |x|\ \ 1\}$
$\|B\|\ \|A\|$

Hence $\|BA\|\ = \|B\|\ \|A\|$

Hence (c) is proved.

Hence the theorem.

Theorem11.3

Let Ω be the set of all invertible linear operators on R^n.

(a) If $A \in \Omega$, $B \in L(R^n)$ and $\|B-A\|\ \|A^{-1}\| < 1$, then $B \in \Omega$.

(b) Ω is an subset of $L(R^n)$ and the mapping A › A^{-1} is continuous on Ω.

[This mapping is also a one-to-one mapping of Ω onto Ω, which is its own inverse.]

Proof

(a) Let $\|A^{-1}\| = 1/\alpha$ and
$\|B-A\| = \beta$.

Then ,

$\|B-A\|\ \|A^{-1}\| < 1$ implies $\beta/\alpha < 1$
i.e $\beta < \alpha$.

For every $x \in R^n$,

$\alpha|x| = \alpha|A^{-1}Ax|$
$\alpha\|A^{-1}\|$
$= |Ax|$

$$= |Ax-Bx+Bx|$$
$$|Ax-Bx|+|Bx|$$
$$|(A-B)(x)|+|Bx|$$
$$\|A-B\|\ |x| +|Bx|$$
$$\beta\,|x| +|Bx|$$

Therefore

$$(\alpha-\beta)|x| \quad |Bx|, \qquad \text{for all } x \in R^n. \qquad \ldots(1)$$

Since $\alpha-\beta > 0$,

if $x \quad 0$, then $Bx \quad 0$

Hence B is one-to-one.

Hence by theorem11.1, $R(B) = R^n$.

Hence B is onto .

Hence B is both one-to-one and onto.

Hence B is invertible,

Hence $B \in \Omega$.

(b) To prove that the mapping $A ›\ A^{-1}$ is continuous on Ω.

Replacing x by $B^{-1}y$ in (1), we get

$$(\alpha-\beta)|\,B^{-1}y\,| \quad |\,B(B^{-1}y)| = |y|, \text{ for all } y \in R^{n.}$$

Therefore,

$$|\,B^{-1}y| \quad \frac{|\,y\,|}{(\alpha-\beta)}$$

Therefore ,

$$\sup\{|\,B^{-1}y| \,/\, y \in R^n \text{ and } |y| \ 1\} \quad \frac{1}{(\alpha-\beta)}$$

Therefore,

$$\|B^{-1}\| \quad \frac{1}{(\alpha-\beta)}$$

Consider $\quad B^{-1} - A^{-1} = B^{-1}A\,A^{-1} - B^{-1}B\,A^{-1}$

$$= B^{-1}(A-B)\,A^{-1}$$

Therefore,

$$\|\,B^{-1}- A^{-1}\| = \|\,B^{-1}(A-B)\ A^{-1}\|$$
$$\|\,B^{-1}\,\|\ \|\,A- B\|\ \|\,A^{-1}\ \|$$
$$\frac{1}{(\alpha-\beta)}(\beta/\alpha)$$

This implies that $\|\,B^{-1}- A^{-1}\| › 0$ as $\beta > 0$ i.e as $\|B - A\| › 0$.

Hence the mapping $A ›\ A^{-1}$ is continuous on Ω.

Hence the theorem.

11.4 Matrix representation of a linear transformation

Suppose $\{x_1, x_2, \ldots x_n\}$ and $\{y_1, y_2, \ldots y_m\}$ are basis of the vector spaces X and Y respectively.

Let $A \in L(X, Y)$.

Since $Ax_j \in Y$ for j=1,2,…n, and

$\{y_1, y_2, \ldots y_m\}$ is a basis of Y,

$$Ax_j = \sum_{i=1}^{m} a_{ij}\, y_i\ , \ j=1,2,\ldots n. \qquad \ldots(1)$$

These numbers can be put into a rectangular array of m rows and n columns, called m by n matrix as follows:

$$[A] = \begin{bmatrix} a_{11} & a_{12} & .. & .. & a_{1n} \\ a_{21} & a_{22} & .. & .. & a_{2n} \\ .. & .. & .. & .. & .. \\ .. & .. & .. & .. & .. \\ a_{m1} & a_{m2} & .. & .. & a_{mn} \end{bmatrix}$$

where the coordinates a_{ij} of the vectors Ax_j appear in the jth column of [A]. Therefore the vectors Ax_j are called column vectors of [A].

With this terminology, the range of A is said to be spanned by the column vectors of [A].

If $x = \sum_{j=1}^{n} c_j x_j$,

$$Ax = A(\sum_{j=1}^{n} c_j x_j)$$

$$= \sum_{j=1}^{n} c_j Ax_j$$

$$= \sum_{j=1}^{n} c_j (\sum_{i=1}^{m} a_{ij} y_i)$$

$$= \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} c_j) y_i$$

Therefore ,

$$Ax = \sum_{i=1}^{m} (\sum_{j=1}^{n} a_{ij} c_j) y_i \qquad \ldots(2).$$

Thus the coordinates of Ax are $\sum_{j=1}^{n} a_{ij} c_j$.

Suppose a m by n matrix is given with real entries a_{ij} and if A is defined as in (2),

then $A \in L(X, Y)$. and

[A] will be the given matrix.

Hence there is a one-to-one correspondence between L(X, Y) and the set of all m by n matrices.

Remark:

[A] depends not only on A but also on the choice of bases in X and Y.

The same A may give rise to many different matrices, if we change bases and vice versa.

11.5 Let us sum up

In this lesson we have seen,

- Definition of vector spaces and some of its properties.
- Definition of linear transformation , sum of two linear transformations and product of two linear transformations

- Definion of norm of a linear transformation
- The set of all linear transformations from R^n to R^m is a linear space
- The set of all linear transformations from R^n to R^m is a metric space
- Definition of operators and inverse operators and
- How to represent a linear transformation in the matrix form.

11.6 Lesson End Activities

1. Write the matrix corresponding to the linear transformations T : R^3 ® |→ R^3 ® give by T (x, y , z) = (3x + z, -2x +y, x+2y+z) with respect to the basis { (1,0,1), (-1,2,1),(2,1,1)} got both the domain & co-domain.

11.7 Reference

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

LESSON – 12

CONTRACTION PRINCIPLE, INVERSE FUNCTION THEOREM AND IMPLICIT FUNCTION THEOREM.

12.0 Introduction
12.1 Aims and objectives
12.2 Contraction Principle
12.2 Inverse function theorem
12.3 Implicit function theorem
12.4 Let us sum up
12.5 Lessson End Activities
12.6 References

12.0 Introduction

In this lesson we are going to study two important theorems in real analysis, inverse function theorem and implicit function theorem. The inverse function theorem states that a continuous differentiable mapping f is invertible in a neighborhood of any point x at which the linear transformation f'(x) is invertible. The implicit function theorem states that if f is continuously differentiable real function in the plane, then the equation f(x, y) =0 can be solved for y in terms of x in a neighborhood of any point (a, b) at which f(a, b)=0 and $\partial f / \partial y$ 0. Likewise , one can solve for x in terms of y near (a, b) if $\partial f / \partial x$ 0.

12.1 Aims and objectives

After studying this lesson, you would know

- The definition of a contraction mapping
- Fixed point theorem
- Continuously differentiable mapping
- Inverse function theorem and
- Implicit function theorem and its linear version

12.2 The contraction Principle.

Definition

Let X be a metric space with metric d.

If φ maps X into X and if there is a number c < 1 such that

$$d(\varphi(x), \varphi(y)) \quad cd(x, y), \qquad \text{for all } x, y \in X,$$

then φ is said to be a contraction of X into X.

Theorem12.1

If X is a complete metric space and if φ is a contraction of X into X, then there exists one and only one point $x \in X$ such that

$$\varphi(x) = x.$$

Proof

Choose $x_0 \in X$ arbitrarily.

Define $\{ x_n \}$ recursively, by setting

$$x_{n+1} = \varphi(x_n), \qquad n=1,2,3,\ldots \qquad \ldots(1)$$

Since φ is a contraction of X into X,

there exists a number c< 1 such that

$$d(\varphi(x), \varphi(y)) \quad cd(x, y), \quad \text{for all } x, y \in X. \qquad \ldots(2)$$

Therefore,

$$d(x_{n+1}, x_n) = d(\varphi(x_n), \varphi(x_{n-1}))$$
$$cd(x_n, x_{n-1}),$$

which in turn gives that

$$d(x_n, x_{n-1}) \quad cd(x_{n-1}, x_{n-2}).$$

Therefore ,

$$d(x_{n+1}, x_n) \quad c^2 d(x_{n-1}, x_{n-2}).$$

Therefore ,by induction,

$$d(x_{n+1}, x_n) \quad c^n d(x_1, x_0), n=0,1,2,\ldots$$

Therefore, if n<m,

$$d(x_m, x_n)\ d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \ldots + d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n)$$
$$= \sum_{i=n+1}^{m} d(x_i, x_{i-1})$$
$$\sum_{i=n+1}^{m} c^{i-1} d(x_1, x_0)$$
$$=(c^n + c^{n+1} + \ldots c^{m-1}) d(x_1, x_0)$$
$$c^n(1 + c + c^2 + \ldots)\, d(x_1, x_0)$$
$$\frac{c^n}{1-c}\, d(x_1, x_0)$$

Since c< 1,

$$\lim_{n›} \quad c^n = 0,$$

Therefore**,**

$$d(x_m, x_n)› \ 0 \text{ as } m, n› \quad .$$

Hence $\{x_n\}$ is a Cauchy sequence in X.

Since X is complete,

every Cauchy sequence of points of X converges to a point in X,

Hence $\{x_n\}$ converges to a point , say x in X.

i.e $\lim_{n›} \quad x_n = x$.

Since φ is a contraction ,

φ is continuous on X.

Hence

$$\varphi(x) = \varphi(\lim_{n›} \quad x_n)$$
$$= \lim_{n›} \quad \varphi(x_n)$$
$$= \lim_{n›} \quad x_{n+1}$$
$$= x.$$

Hence there exists a point $x \in X$ such that $\varphi(x) = x$.

Hence the theorem.

12.2 Inverse function theorem

Definition

Suppose E is an open subset in R^n, **f** maps E into R^m and $x \in E$.

f is said to be differentiable at **x**,

if there exists a linear transformation A of R^n into R^m such that

$$\lim_{h› 0} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0$$

and we write **f'(x)** =A.

If **f** is differentiable at every **x** $\in$ E, then **f** is said to differentiable in E.

Definition

A differentiable mapping **f** of an open set $E \subset R^n$ into R^m is said to be continuously differentiable in E if **f'** is a continuous mapping of E into $L(R^n, R^m)$
i.e if for every x $\in$ E and for every $\varepsilon > 0$,
there exists a $\delta > 0$ such that
$\|\mathbf{f}'(\mathbf{y}) - \mathbf{f}'(\mathbf{x})\| < \varepsilon$, if **y** $\in$ E and $|\mathbf{x}-\mathbf{y}| < \delta$,
then **f** is said to be C' mapping or **f** $\in$ C'(E).

Theorem 12.2 Inverse function theorem

Suppose **f** is a C'-mapping of an open subset $E \subset R^n$ into R^n,
f'(a) is invertible for some **a** $\in$ E and **b=f'(a).** Then
(a) there exist open sets U and V in R^n such that **a** $\in$ U, **b** $\in$ V,
f is one-to-one on U and **f**(U) = V.
(b)If **g** is the inverse of **f**, defined in V by
g(f(x))=x , **x** $\in$ U,
then **g** $\in$ C'(V).

[Writing the equation **y=f(x)** in component form, the conclusion of the theorem can be interpreted as follows:
The system of n equations
$y_i = f_i(x_1, x_2, \ldots . x_n)$ i=1,2,…n
can be solved for $x_1, x_2, \ldots . x_n$ in terms of $y_1, y_2, \ldots . y_n$.
If we restrict **x** and **y** to be small enough neighborhoods of **a** and **b**, the solutions are unique and continuously differentiable.]

Proof

(a) Let **f'(a)** =A.
Choose λ such that
$2\lambda \|A^{-1}\| = 1$ …(1)
Since **f** is a C'-mapping on E and **a** $\in$ E,
f' is continuous at **a**.
Hence there exists an open ball U $\subset$ E, with centre at **a** such that
$\|\mathbf{f}'(\mathbf{x}) - A\| < \lambda$, for all **x** $\in$ U. …(2)

To each **y** $\in$ R^n, we associate a function φ on E as
$\varphi(\mathbf{x}) = \mathbf{x} + A^{-1}(\mathbf{y} - \mathbf{f}(\mathbf{x}))$, **x** $\in$ E …(3)
so that if **f(x)=y,** then $\varphi(\mathbf{x}) = \mathbf{x}$.
Conversely ,
if $\varphi(\mathbf{x}) = \mathbf{x}$,
then $A^{-1}(\mathbf{y} - \mathbf{f}(\mathbf{x})) = 0$
implies $\mathbf{y} - \mathbf{f}(\mathbf{x}) = 0$
implies $\mathbf{y} = \mathbf{f}(\mathbf{x})$
Hence **f(x) = y** if and only if $\varphi(\mathbf{x}) = \mathbf{x}$
i.e **x** is a fixed point of φ. …(4)
Now $\varphi'(\mathbf{x}) = I - A^{-1}\mathbf{f}'(\mathbf{x})$
$= A^{-1}A - A^{-1}\mathbf{f}'(\mathbf{x})$
$= A^{-1}(A - \mathbf{f}'(\mathbf{x}))$

Therefore for $\mathbf{x} \in U$,

$$\| \varphi'(\mathbf{x}) \| = \| A^{-1}(A - \mathbf{f'(x)}) \|$$
$$\| A^{-1} \| \, \|A - \mathbf{f'(x)}) \|$$
$$(1/\, 2\lambda \;)\, \lambda \quad \text{[from (1) and (2)]}$$

Therefore,

$$\| \varphi'(\mathbf{x}) \| \quad ½ \qquad \text{for all } \mathbf{x} \in U.$$

Therefore by the result,

"Suppose **f** maps a convex open set $E \subset R^n$ into R^m.

If **f** is differentiable in E, and there is a real number M such that

$$\|\mathbf{f'(x)}\| \quad M \qquad \text{for every } \mathbf{x} \in E.$$

Then $|\mathbf{f(b)} - \mathbf{f(a)}| \quad M \; |\mathbf{b-a}|$ for all $\mathbf{a} \in E, \mathbf{b} \in E$.",

we get that ,

$$|\varphi(\mathbf{x_1}) - \varphi(\mathbf{x_2})| \quad (1/2)\, |\mathbf{x_1} - \mathbf{x_2}|, \qquad \text{for all } \mathbf{x_1}, \mathbf{x_2} \in U. \qquad \ldots(5)$$

Therefore,

φ is a contraction of U into U.

Therefore ,

φ has atmost one fixed point in U. [by theorem12.1]

Therefore ,

$\mathbf{f(x)} = \mathbf{y}$ for atmost one $\mathbf{x} \in U$. [From (4)]

Hence **f** is one-to-one in U.

Let $V = \mathbf{f}(U)$.

To prove that V is open.

Let $\mathbf{y_0} \in V$.

Then $\mathbf{y_0} = \mathbf{f(x_0)}$ for some $\mathbf{x_0} \in U$.

Let B be the open ball with centre at $\mathbf{x_0}$ and radius $r > 0$, so small that its closure $\overline{B}$ lies in U.

Claim: $\mathbf{y} \in V$ whenever $|\mathbf{y} - \mathbf{y_0}| < \lambda r$.

Let **y** be such that $|\mathbf{y} - \mathbf{y_0}| < \lambda r$.

From (3), $|\varphi(\mathbf{x_0}) - \mathbf{x_0}| = |A^{-}1(\mathbf{y} - \mathbf{y_0})|$

$$\| A^{-1} \| \, |\mathbf{y} - \mathbf{y_0}|$$
$$< \| A^{-1} \| \, \lambda \, r$$
$$= (1/2) r \qquad \text{[from (1)]}$$

Hence,

$$|\varphi(\mathbf{x_0}) - \mathbf{x_0}| < r/2. \qquad \ldots(6)$$

If $\mathbf{x} \in B$, then by (5),

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{x_0})| \quad (1/2)\, |\mathbf{x} - \mathbf{x_0}| \qquad \ldots(7)$$

From (6) and (7), we get that

whenever $\mathbf{x} \in B$,

$$|\varphi(\mathbf{x}) - \mathbf{x_0}| = |\varphi(\mathbf{x}) - \varphi(\mathbf{x_0}) + \varphi(\mathbf{x_0}) - \mathbf{x_0}|$$
$$|\varphi(\mathbf{x}) - \varphi(\mathbf{x_0})| + |\varphi(\mathbf{x_0}) - \mathbf{x_0}|$$
$$< (1/2)\, |\mathbf{x} - \mathbf{x_0}|$$
$$(r/2) + (r/2) = r.$$

Hence ,

$$|\varphi(\mathbf{x}) - \mathbf{x_0}| \quad r \qquad \text{whenever } \mathbf{x} \in B.$$

This implies that $\varphi(\mathbf{x}) \in B$.

Hence φ maps B into B.

Since φ is a contraction of U into U and $\overline{B} \subset U$,

φ is a contraction of $\overline{B}$ into $\overline{B}$.

Since $\overline{B}$ is a closed subset of R^n and R^n is complete, $\overline{B}$ is complete.

Hence by the theorem12.1,

φ has a fixed point $\mathbf{x} \in \overline{B}$.

i.e $\varphi(\mathbf{x}) = \mathbf{x}$.

For this $\mathbf{x}$, $f(\mathbf{x}) = \mathbf{y}$. [From (4)]

Hence,

$$\mathbf{y} \in \mathbf{f}(\overline{B}) \subset \mathbf{f}(U) = V.$$

Hence ,

$$\mathbf{y} \in V \text{ whenever } |\mathbf{y} - \mathbf{y_0}| < \lambda r.$$

Hence V is open.

Hence part(a) of the theorem is proved.

To prove part(b):

Let $\mathbf{y} \in V$, $\mathbf{y+t} \in V$.

Since $V=\mathbf{f}(U)$,

there exists a $\mathbf{x} \in U$, $\mathbf{x+h} \in U$ such that

$$\mathbf{y} = \mathbf{f(x)} \text{ and}$$
$$\mathbf{y+k} = \mathbf{f(x+h)}.$$

From (3),

$$\begin{aligned}\varphi(\mathbf{x+h}) - \varphi(\mathbf{x}) &= [(\mathbf{x+h}) + A^{-1}(\mathbf{y - f(x+h)})] - [\mathbf{x} + A^{-1}(\mathbf{y - f(x)})] \\ &= \mathbf{h} + A^{-1}(\mathbf{f(x)} - \mathbf{f(x+h)}) \\ &= \mathbf{h} - A^{-1}\mathbf{k} \qquad \ldots(8)\end{aligned}$$

By(5),

$$\begin{aligned}|\varphi(\mathbf{x+h}) - \varphi(\mathbf{x})| &\quad (1/2)\,|(\mathbf{x+h}) - \mathbf{x}| \\ &= (1/2)|\mathbf{h}|, \qquad \ldots(9)\end{aligned}$$

From (8) and (9), we get

$$\begin{aligned}|\mathbf{h} - A^{-1}\mathbf{k}| &= |\varphi(\mathbf{x+h}) - \varphi(\mathbf{x})| \\ &\quad (1/2)|\mathbf{h}|\end{aligned}$$

Therefore ,

$$\begin{aligned}|A^{-1}\mathbf{k}| &= |\mathbf{h} - (\mathbf{h} - A^{-1}\mathbf{k})| \\ &\quad |\mathbf{h}| - |\mathbf{h} - A^{-1}\mathbf{k}| \\ &\quad |\mathbf{h}| - |\mathbf{h}|/2 = |\mathbf{h}|/2\end{aligned}$$

Hence $|A^{-1}\mathbf{k}| \quad |\mathbf{h}|/2$

Therefore ,

$$\begin{aligned}|\mathbf{h}| &\quad 2|A^{-1}\mathbf{k}| \\ &\quad 2\|A^{-1}\|\,|\mathbf{k}| \\ &\quad (1/\lambda)|\mathbf{k}| \qquad \ldots(10)\end{aligned}$$

Therefore ,

$$\frac{1}{|k|} \leq \frac{1}{\lambda |h|} \qquad \ldots(11)$$

Let **g** be the inverse of **f**, defined in V by

$$\mathbf{g(f(x))=x}, \quad \mathbf{x} \in U.$$

Let $\mathbf{x} \in U$.

From (2) ,

$$\|\mathbf{f'(x)} - A\| < \lambda = 1/\|A^{-1}\|$$

i.e $\|\mathbf{f'(x)} - A\| < 1/\|A^{-1}\|$, where $A = \mathbf{f'(a)}$ is invertible.

Hence by theorem 11.3

f'(x) is invertible.
Let the inverse of **f'(x)** be T.

Consider $\mathbf{g(y+k) - g(y)} - T\mathbf{k} = \mathbf{g(f(x+h)) - g(f(x))} - T\mathbf{k}$

$= \mathbf{x+h - x} - T\mathbf{k}$

$= \mathbf{h} - T\mathbf{k}$

$= T\mathbf{f'(x)h} - T\mathbf{k}$

[since T is the inverse of **f'(x),** T**f'(x)=I**]

$= -T\,[\mathbf{k - f'(x)h}]$

$= -T\,[\mathbf{f(x+h) - f(x) - f'(x)h}]$

Therefore,

$$|\mathbf{g(y+k) - g(y)} - T\mathbf{k}| = |\,T\,[\mathbf{f(x+h) - f(x) - f'(x)h}]|$$

$$\|T\|\ |\mathbf{f(x+h) - f(x) - f'(x)h}| \qquad \ldots(12)$$

From (11) and (12),

$$\frac{|g(y+k)-g(y)-Tk|}{|k|} \quad \frac{\|T\||\ f(x+h)-f(x)-f'(x)h|}{\lambda\,|h|} \qquad \ldots(13)$$

From (10), it is clear that $\mathbf{h} \to \mathbf{0}$ as $\mathbf{k} \to \mathbf{0}$.
Also, since **f** is differentiable at **x**,

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|f(x+h)-f(x)-f'(x)h|}{|h|} = 0.$$

Therefore from(13),

$$\lim_{\mathbf{k}\to\mathbf{0}} \frac{|g(y+k)-g(y)-Tk|}{|k|} = 0.$$

This implies that

g'(y) exists and

g'(y) = T.

Since T is the inverse of **f'(x) = f'(g(y)),**

$$\mathbf{g'(y)} = \{\mathbf{f'(g(y))}\}^{-1} \quad \text{for all } \mathbf{y} \in V.$$

Hence **g** is differentiable on V.
Hence **g** is a continuous mapping of V onto U.
Also **f'** is a continuous mapping U into the set Ω of all invertible elements of L(R^n).
Also the inversion is a continuous mapping of Ω onto Ω.
Hence **g'** is a continuous mapping on V.
Hence **g** $\in$ C'(V).
Hence the theorem.

Theorem12.3

If **f** is a C'- mapping of an open set E $\subset R^n$ into R^m and if **f'(x)** is invertible for every **x** $\in$ E, then **f**(W) is an open subset of R^n for every open set W $\subset$ E.
In other words, **f** is an open mapping of E into R^n.

Proof

Let W $\subset$ E is an open set.
To prove that **f**(W) is an open subset of R^n.
Let **b** $\in$ **f**(W).
Then there exists a **a** $\in$ W such that **b = f(a).**
Since **f'(x)** is invertible for every **x** $\in$ E, and **a** $\in$ W $\subset$ E,
f'(a) is invertible.
Hence by part (a) of Theorem12.2,

there exist open sets $U_{\mathbf{a}} \subset W$ and $V_{\mathbf{b}} \subset \mathbf{f}(W)$
such that $\mathbf{a} \in U$, $\mathbf{b} \in V$, $\mathbf{f}$ is one-to-one on U and $\mathbf{f}(U) = V$.
Hence $\mathbf{f}(W)$ can be written as $\mathbf{f}(W) = \bigcup_{b \in f(W)} V_b$.
Sine union of open sets is open, $\mathbf{f}(W)$ is open.
Hence the theorem.

12.3 Implicit function theorem

Notation

If $\mathbf{x} = (x_1, x_2, \ldots x_n) \in R^n$ and
$\mathbf{y} = (y_1, y_2, \ldots y_m) \in R^m$,
We write,
$(\mathbf{x}, \mathbf{y}) = (x_1, x_2, \ldots x_n, y_1, y_2, \ldots y_m) \in R^{n+m}$.
The first entry in $(\mathbf{x}, \mathbf{y})$ or in a similar symbol will always be a vector in R^n, the second will be a vector in R^m.

Every vector $A \in L(R^{n+m}, R^n)$ can be split into two linear transformation A_x and A_y, defined by

$$A_x\mathbf{h} = A(\mathbf{h}, \mathbf{0}),$$
$$A_y\mathbf{k} = A(\mathbf{0}, \mathbf{k}), \quad \text{for any } \mathbf{h} \in R^n, \mathbf{k} \in R^m.$$

Then $A_x \in L(R^n)$, $A_y \in L(R^m)$ and

$$\begin{aligned} A(\mathbf{h}, \mathbf{k}) &= A[(\mathbf{h}, \mathbf{0}) + (\mathbf{0}, \mathbf{k})] \\ &= A(\mathbf{h}, \mathbf{0}) + A(\mathbf{0}, \mathbf{k}) \qquad \text{[since A is linear]} \\ &= A_x\mathbf{h} + A_y\mathbf{k}. \end{aligned}$$

Theorem12.4 [Linear version of the implicit function theorem]

If $A \in L(R^{n+m}, R^n)$ and if A_x is invertible,
then there corresponds to every $\mathbf{k} \in R^m$ a unique $\mathbf{h} \in R^n$ such that $A(\mathbf{h}, \mathbf{k}) = \mathbf{0}$.
This $\mathbf{h}$ can be computed from $\mathbf{k}$ by the formula
$\mathbf{h} = -(A_x)^{-1} A_y\mathbf{k}$.

Proof

Now $A(\mathbf{h}, \mathbf{k}) = \mathbf{0}$ if and only if $A_x\mathbf{h} + A_y\mathbf{k} = \mathbf{0}$.
Since A_x is invertible,

$(A_x)^{-1}$ exists.

Therefore premultiplying $A_x\mathbf{h} + A_y\mathbf{k} = \mathbf{0}$ by $(A_x)^{-1}$, we get

$$(A_x)^{-1}(A_x\mathbf{h} + A_y\mathbf{k}) = \mathbf{0}$$
$$\mathbf{h} + (A_x)^{-1} A_y\mathbf{k} = \mathbf{0}.$$

Therefore ,

$$\mathbf{h} = -(A_x)^{-1} A_y\mathbf{k}.$$

Hence the theorem.

Check your progress

Prove that A_x and A_y are linear.

Theorem 12.5 The implicit function theorem

Let $\mathbf{f}$ be a C'-mapping of an open set $E \subset R^{n+m}$ into R^n, such that
$\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$ for some point $(\mathbf{a}, \mathbf{b}) \in E$.
Put $A = \mathbf{f}'(\mathbf{a}, \mathbf{b})$ and assume that A_x is invertible.

Then there exist open sets $U \subset R^{n+m}$ and $W \subset R^{m}$, with $(\mathbf{a}, \mathbf{b}) \in U$ and $\mathbf{b} \in W$, having the following property:

To every $\mathbf{y} \in W$ corresponds a unique $\mathbf{x}$ such that

$$(\mathbf{x}, \mathbf{y}) \in U \quad \text{and} \quad \mathbf{f(x, y)=0}. \qquad \ldots(1)$$

If this $\mathbf{x}$ is defined to be $\mathbf{g(y)}$,

then $\mathbf{g}$ is a C'-mapping of W into R^{n},

$$\mathbf{g(b) = a},$$

$$\mathbf{f(g(y), y) = 0} \quad \mathbf{y} \in W. \qquad \ldots(2)$$

and $$\mathbf{g'(b) = -} (A_x)^{-1} A_y. \qquad \ldots(3)$$

The function g is implicitly defined by (2).

Hence the name of the theorem.

Proof

Define $\mathbf{F}: E › R^{n+m}$ by

$$\mathbf{F(x, y) =(f(x, y), y)} \quad \text{for} \quad (\mathbf{x}, \mathbf{y}) \in E.$$

Since $\mathbf{f}$ is a C'-mapping,

$\mathbf{F}$ is a C'-mapping of E into R^{n+m}.

Claim: $\mathbf{F'(a, b)}$ is an invertible element of $L(R^{n+m})$

Since $\mathbf{f'(a, b)}$ exists and equals A,

$$\mathbf{f(a+h, b+k) = f(a, b)+} A(\mathbf{h, k}) \mathbf{+r(h, k)},$$

where $\mathbf{r}$ is the remainder that occurs in the definition of $\mathbf{f'(a, b)}$.

Therefore,

$$\mathbf{f(a+h, b+k) =} A(\mathbf{h, k}) \mathbf{+r(h, k)}. \quad [\text{Since } \mathbf{f(a, b)=0}] \quad \ldots(4)$$

Consider
$$\begin{aligned}\mathbf{F(a+h, b+k) - F(a, b)} &= \mathbf{(f(a+h, b+k), b+k) - (f(a, b), b)}\\ &= \mathbf{(f(a+h, b+k), b+k) - (0, b)}\\ &= \mathbf{(f(a+h, b+k), k)}\\ &= \mathbf{(} A(\mathbf{h, k}) \mathbf{+r(h, k),\ k+0)} \qquad [\text{from } (4)]\\ &= (A(\mathbf{h, k}), \mathbf{k}) + (\mathbf{r(h, k), 0})\end{aligned}$$

Therefore ,

$\mathbf{F'(a, b)}$ is the linear operator on R^{n+m} that maps $(\mathbf{h, k})$ to $(A(\mathbf{h, k}), \mathbf{k})$.

If this image vector is $\mathbf{0}$,

then $A(\mathbf{h, k}) = \mathbf{0}$ and $\mathbf{k=0}$.

Hence $A(\mathbf{h, 0})=\mathbf{0}$.

This implies $\mathbf{h=0}$. [By theorem12.4]

Therefore if the image vector of $\mathbf{F'(a, b)}$ at $(\mathbf{h, k})$ is $\mathbf{0}$,

then both $\mathbf{h=0}$ and $\mathbf{k=0}$.

Hence $\mathbf{F'(a, b)}$ is one-to-one.

Hence by theorem 11.1,

$\mathbf{F'(a, b)}$ is onto.

Hence $\mathbf{F'(a, b)}$ is both one-to-one and onto.

Hence $\mathbf{F'(a, b)}$ is invertible.

Hence our claim is proved.

Hence inversion theorem can be applied to **F.**

Hence by part(a) of the inversion theorem,

there exist open sets U and V in R^{n+m} with $(\mathbf{a, b}) \in U$, $(\mathbf{0, b}) \in V$, such that

F is a one-to-one mapping of U onto V.

Let $W =\{\mathbf{y} \in R^{m} / \ (\mathbf{0, y}) \in V\}$

Therefore $\mathbf{b} \in W$. [since $(\mathbf{0, b}) \in V$].

Since V is open,

W is also open.

If $\mathbf{y} \in W$,

then $(\mathbf{0}, \mathbf{y}) \in V = \mathbf{F}(U)$.

Hence there exists a point $(\mathbf{x}, \mathbf{y})$ in R^{n+m} such that

$$(\mathbf{0}, \mathbf{y}) = \mathbf{F}(\mathbf{x}, \mathbf{y}) .$$

Therefore ,

$$(\mathbf{0}, \mathbf{y}) = (\mathbf{f}(\mathbf{x}, \mathbf{y}), \mathbf{y})$$

This implies that

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}, \text{ for this } \mathbf{x}.$$

Suppose for the same $\mathbf{y}$, there exists another $\mathbf{x'}$ in R^n, such that

$$\mathbf{f}(\mathbf{x'}, \mathbf{y}) = \mathbf{0}.$$

Then

$$\begin{aligned} \mathbf{F}(\mathbf{x'}, \mathbf{y}) &= (\mathbf{f}(\mathbf{x'}, \mathbf{y}), \mathbf{y}) \\ &= (\mathbf{0}, \mathbf{y}) \\ &= (\mathbf{f}(\mathbf{x}, \mathbf{y}), \mathbf{y}) \\ &= \mathbf{F}(\mathbf{x}, \mathbf{y}). \end{aligned}$$

Since $\mathbf{F}$ is one-to-one in U,

this implies that $\mathbf{x'} = \mathbf{x}$.

Hence this $\mathbf{x}$ is unique.

Hence there exist open sets $U \subset R^{n+m}$ and $W \subset R^m$, with $(\mathbf{a}, \mathbf{b}) \in U$ and $\mathbf{b} \in W$, with the property that:

To every $\mathbf{y} \in W$ corresponds a unique $\mathbf{x}$ such that

$$(\mathbf{x}, \mathbf{y}) \in U \qquad \text{and}$$

$$\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}.$$

Hence first part of the theorem is proved.

To prove the second part:

As we have seen in the first part,

to every $\mathbf{y} \in W$ there is a unique $\mathbf{x}$ such that

$$(\mathbf{x}, \mathbf{y}) \in U \qquad \text{and } \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}.$$

Define this $\mathbf{x}$ to be $\mathbf{g}(\mathbf{y})$ for every $\mathbf{y} \in W$.

Then $\quad \mathbf{g}(\mathbf{b}) = \mathbf{a}. \qquad$ [since $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$]

Also $\quad (\mathbf{g}(\mathbf{y}), \mathbf{y}) = (\mathbf{x}, \mathbf{y}) \in U$ and

$$\mathbf{f}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{0}. \qquad \ldots(5)$$

Therefore ,

$$\mathbf{F}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = (\mathbf{f}(\mathbf{g}(\mathbf{y}), \mathbf{y}), \mathbf{y}) = (\mathbf{0}, \mathbf{y}), \text{ for all } \mathbf{y} \in W. \qquad \ldots(6)$$

Let $\mathbf{G}$ be the mapping of V into U that inverts $\mathbf{F}$ i.e $\mathbf{G} = \mathbf{F}^{-1}$.

Also $\mathbf{F} \in C'$, implies $\mathbf{G} \in C'$

Then from(6),

$$\mathbf{G}[\mathbf{F}(\mathbf{g}(\mathbf{y}), \mathbf{y})] = \mathbf{G}(\mathbf{0}, \mathbf{y}) , \text{ for all } \mathbf{y} \in W.$$

$$\textbf{i.e} \qquad (\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{G}(\mathbf{0}, \mathbf{y}), \text{ for all } \mathbf{y} \in W. \qquad \ldots(7)$$

Since $\mathbf{G} \in C'$, (7) shows that $\mathbf{g} \in C'$.

To compute $\mathbf{g'}(\mathbf{b})$:

Let $\quad (\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{\Phi}(\mathbf{y})$.

Then $\quad \mathbf{\Phi'}(\mathbf{y})\,\mathbf{k} = (\mathbf{g'}(\mathbf{y})\mathbf{k}, \mathbf{y}\mathbf{k})$ for all $\mathbf{y} \in W$ and $\mathbf{k} \in R^m$. $\qquad \ldots(8)$

From (5),

$$\mathbf{f}(\mathbf{\Phi}(\mathbf{y})) = \mathbf{f}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{0}.$$

Therefore by chain rule,

$$\mathbf{f'(\Phi(y))\,\Phi'(y) = 0.} \qquad \ldots(9)$$

When **y=b,**

$$\begin{aligned}\mathbf{\Phi(y)} &= \mathbf{\Phi(b)}\\ &= \mathbf{(g(b), b)}\\ &= \mathbf{(a, b)}\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbf{f'(\Phi(y))} &= \mathbf{f'(\Phi(b))}\\ &= \mathbf{f'(a, b)}\\ &= A.\end{aligned}$$

Substituting in (9), we get

$$A\mathbf{\Phi'(b)} = \mathbf{0.} \qquad \ldots(10)$$

We know that for all $\mathbf{h} \in R^n$, $\mathbf{k} \in R^m$,

$$A(\mathbf{h}, \mathbf{k}) = A_x\mathbf{h} + A_y\mathbf{k},$$

where $A_x\mathbf{h} = A(\mathbf{h}, \mathbf{0})$ and

$A_y\mathbf{k} = A(\mathbf{0}, \mathbf{k})$.

Therefore

$$\begin{aligned}A_x\mathbf{g'(b)k} + A_y\mathbf{k} &= A(\mathbf{g'(b)k}, \mathbf{k})\\ &= A\mathbf{\Phi'(b)k} && \text{[from (8)]}\\ &= \mathbf{0}, \quad \text{for every } \mathbf{k} \in R^m. && \text{[from(10)]}\end{aligned}$$

Therefore

$$A_x\mathbf{g'(b)} + A_y = \mathbf{0}.$$

Therefore

$$\mathbf{g'(b)} = -(A_x)^{-1}A_y.$$

Hence the theorem.

Remark

The equation **f(x, y) =0** can be written as a system of n equations in n+m variables:

$$\begin{aligned}&f_1(x_1, x_2, \ldots x_n, y_1, y_2, \ldots y_m) = 0\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \ldots(1)\\ &f_n(x_1, x_2, \ldots x_n, y_1, y_2, \ldots y_m) = 0.\end{aligned}$$

The assumption that A_x is invertible means that the n by n matrix

$$\begin{bmatrix} D_1 f_1 & .. & .. & D_n f_1\\ .. & .. & .. & ..\\ .. & .. & .. & ..\\ D_1 f_n & .. & .. & D_n f_n \end{bmatrix}$$

evaluated at **(a, b)** defines an invertible linear operator in R^n.

In other words, its column vectors should be independent i.e its determinant should not be equal to zero.

If (1) holds when **x=a, y=b,** then the conclusion of the theorem is that:

(1) can be solved for $x_1, x_2, \ldots x_n$ in terms of $y_1, y_2, \ldots y_m$. for every **y** near **b**, and that these solutions are continuously differentiable functions of **y**.

Example

Take n=2, m=3.

Consider the mapping $\mathbf{f}=(f_1, f_2)$ of R^5 into R^2 given by

$$f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1} + x_2y_1 - 4y_2 + 3$$

$$f_2(x_1, x_2, y_1, y_2, y_3) = x_2\cos x_1 - 6x_1 + 2y_1 - y_3.$$

Take **a**=(0, 1) and **b**=(3,2,7),
then **f(a, b)**=0.

Now $D_1f_1(x_1, x_2, y_1, y_2, y_3) = 2e^{x_1}$

Therefore ,

$$D_1f_1(0, 1, 3, 2, 7) = 2$$

Similarly ,

$$\begin{aligned}
D_2f_1(0, 1, 3, 2, 7) &= 3\\
D_3f_1(0, 1, 3, 2, 7) &= 1\\
D_4f_1(0, 1, 3, 2, 7) &= -4\\
D_5f_1(0, 1, 3, 2, 7) &= 0\\
D_1f_2(0, 1, 3, 2, 7) &= -6\\
D_2f_2(0, 1, 3, 2, 7) &= 1\\
D_3f_2(0, 1, 3, 2, 7) &= 2\\
D_4f_2(0, 1, 3, 2, 7) &= 0\\
D_5f_2(0, 1, 3, 2, 7) &= -1.
\end{aligned}$$

Therefore the matrix of the transformation A=**f'(a, b)** is

$$[A] = \begin{bmatrix} 2 & 3 & 1 & -4 & 0 \\ -6 & 1 & 2 & 0 & -1 \end{bmatrix}.$$

Hence,

$$[A_x] = \begin{bmatrix} 2 & 3 \\ -6 & 1 \end{bmatrix},$$

And $$[A_y] = \begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

Since the column vectors of $[A_x]$ i.e $\begin{bmatrix} 2 \\ -6 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ are independent,

A_x is invertible.

Therefore by the implicit function theorem ,

a C'-mapping **g** exists in a neighborhood of **b**=(3,2,7) such that

g(b) = g(3, 2, 7)=(0, 1)**=a**

and **f(g(y), y) =0.**

$$[(A_x)^{-1}] = \frac{1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}$$

Moreover,

from **g'(b) =-**$(A_x)^{-1}A_y$, we get

$$\mathbf{g'}(3, 2, 7) = -\frac{1}{20}\begin{bmatrix} 1 & -3 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} 1 & -4 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{5} & -\frac{3}{20} \\ -\frac{1}{2} & \frac{6}{5} & \frac{1}{10} \end{bmatrix}.$$

In terms of partial derivatives,
the conclusion is:
At the point (3, 2, 7),

D_1g_1 = ¼ D_2g_1 = 1/5 D_3g_1=-3/20
D_1g_2 = -1/2 D_2g_2 = 6/5 D_3g_1 = 1/10.

12.4Let us sum up

In this lesson we have studied

- Contraction mapping
- Fixed point theorem
- Inverse function theorem
- Implicit function theorem and
- To compute the derivative of the inverse function at a given point.

12.5 Lesson End Acitivites

Let **f**= $(f_{1,} f_2)$ be the mapping of R^2 into R^2 given by

$f_1(x, y) = e^x\cos y$,

$f_2(x, y) = e^x\sin y$.

Put **a**=(0, π/3), **b**=f(**a**).

1.Find an explicit formula for **g**.

2.Compute **f'(a)** and **g'(b)**

3.Verify the formula **g'(y) = { f'(g(y))}**$^{-1}$ when **y=b.**

12.6 References

1. Principles of Mathematical Analysis by Walter Rudin.

Lesson 13 Determinants

13.0 Introduction

Determinants are numbers associated to square matrices, and hence to the operators represented by such matrices. In this lesson we are going to study the definition of determinants and some of its properties.

13.1 Aims and objectives

After studying this lesson, you would know

- The general definition of the determinant
- Important properties of the determinant
- Jacobians

13.2 Determinants

Definition

If $(j_1, j_2, \dots j_n)$ is an ordered n-tuple of integers, the function $s(j_1, j_2, \dots j_n)$ is defined as

$$s(j_1, j_2, \dots j_n) = \prod_{p<q} \operatorname{sgn}(j_q - j_p),$$

where
$\operatorname{sgn} x = 1$ if $x > 0$
$\operatorname{sgn} x = -1$ if $x < 0$
$\operatorname{sgn} x = 0$ if $x = 0$.

Thus $s(j_1, j_2, \dots j_n) = 1, -1$, or 0,
and it changes sign if any two of the j's are interchanged.

Definition

Let [A] be the matrix of a linear operator A on R^n, relative to the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots \mathbf{e}_n\}$, with entries a(I, j) in the ith row and jth column.

The determinant of [A] is defined to be the number

$$\det [A] = \quad s(j_1, j_2, \dots j_n) a(1, j_1) a(2, j_2) \dots . a(n, j_n).$$

The sum extends over all ordered n-tuples of integers $(j_1, j_2, \dots j_n)$ with $1 \quad j \quad n$.

The column vectors $\mathbf{x}_j$ of [A] are

$$\mathbf{x_j} = \sum_{i=1}^{n} a(i, j) e_i \quad (1 \quad j \quad n)$$

It will be convenient to think of det [A] as a function of the column vectors of [A]. If we write

$$\det [A] = \det (\mathbf{x_1}, \mathbf{x_2}, \dots . \mathbf{x_n}),$$

then det is now a real function on the set of all ordered n-tuples of vectors in R^n.

Theorem13.1

(a) If **I** is the identity operator on R^n, then det [**I**] = det ($\mathbf{e_1}, \mathbf{e_2}, \ldots \mathbf{e_n}$) = 1.

(b) det is a linear function of each of the column vectors $\mathbf{x}_j$, if the others are held fixed.

(c) If $[A]_1$ is obtained from [A] by interchanging two columns, then

det $[A]_1$ = - det [A].

(d) If [A] has two equal columns,

a. then det [A] =0.

Proof

(a) If A= **I**,

then a(I, i) =1 and

a(I, j) = 0 for i j .

Hence,

det [I] = s(1, 2, 3, …n) =1

Hence,

det [**I**] = det ($\mathbf{e_1}, \mathbf{e_2}, \ldots \mathbf{e_n}$) = 1.

Hence part(a) of the theorem is proved.

(b) By definition , s($j_1, j_2, \ldots j_n$) = 0 if any two of the j's are equal.

Each of the remaining n! products in

det [A] = s($j_1, j_2, \ldots j_n$)a(1, j_1)a(2, j_2)….a(n, j_n),

contains exactly one factor from each column.

Hence det is a linear function of each of the column vectors $\mathbf{x}_j$, if the others are held fixed.

(c) Let $[A]_1$ is obtained from [A] , by interchanging two columns.

Since s($j_1, j_2, \ldots j_n$) = $\prod_{p<q} \mathrm{sgn}(j_q - j_p)$,

changes sign if any two of the j's are interchanged,

det $[A]_1$ = - det [A].

(d) Let $[A]_1$ is obtained from [A] , by interchanging two columns.

Then by I,

det $[A]_1$ = - det [A]. …..(1)

If the two columns are equal,

then $[A]_1$ = [A].

Therefore ,

det $[A]_1$ = det[A]. …(2)

From (1) and (2), we get that

det [A] = 0.

Hence the theorem.

Theorem13.2

If [A] and [B] are n by n matrices, then

det ([B][A]) =det [B] det [A].

Proof

Let $\mathbf{x_1}, \mathbf{x_2}, \ldots \mathbf{x_n}$ are the columns of [A].

Then $\mathbf{x_j} = \sum_{i=1}^{n} a(i, j)e_i$, for j=1, 2, …n. …(1)

Define $\Delta_B(\mathbf{x_1}, \mathbf{x_2}, \ldots \mathbf{x_n}) = \Delta_B[A]$ = det ([B] [A]). …(2)

The columns of [B][A] are

$$B\mathbf{x_1},\ B\mathbf{x_2},\ \ldots..\ B\mathbf{x_n}.$$

Therefore,

$$\Delta_B(\mathbf{x_1, x_2, \ldots x_n}) = \det(B\mathbf{x_1}, B\mathbf{x_2}, \ \ldots..\ B\mathbf{x_n}).$$

Hence Δ_B has all the properties (b) to (d) of theorem13.1.

From (1),

$$\mathbf{x_1} = \sum_{i=1}^{n} a(i,1)e_i\ .$$

Therefore using property (b) of theorem13.1, we get

$$\Delta_B[A] = \Delta_B\Big(\sum_{i=1}^{n} a(i,1)e_i\ ,\ \mathbf{x_2, \ldots x_n}\Big)$$

$$= \sum_{i=1}^{n} a(i,1)\ \Delta_B(\mathbf{e_i, x_2, \ldots x_n}).$$

Repeating this for the column vectors $\mathbf{x_2, \ldots x_n}$, we get

$$\Delta_B[A] = \quad a(i_1,1)a(i_2, 2)\ldots.a(i_n, n)\ \Delta_B(\ e_{i_1}, e_{i_2}, \ldots.e_{i_n}\ ), \qquad \ldots(3)$$

The sum being extended over all ordered n-tuples ($i_1, i_2, \ldots.i_n$) with 1 i_r n.

By (c) and (d) of theorem13.1, we get

$$\Delta_B(\ e_{i_1}, e_{i_2}, \ldots.e_{i_n}\ ) = t(\ i_1, i_2, \ldots.i_n)\ \Delta_B(\mathbf{e_1, e_2, \ldots e_n}),$$

where t= 1, 0, -1.

Since [B][I] = [B],

from(2), we get that

$$\Delta_B(\mathbf{e_1, e_2, \ldots e_n}) = \Delta_B[I]$$
$$= \det(\ [B][I])$$
$$= \det\,[B].$$

Therefore ,

$$\Delta_B(\ e_{i_1}, e_{i_2}, \ldots.e_{i_n}\ ) = t(\ i_1, i_2, \ldots.i_n)\ \det\,[B].$$

Substituting this in(3), we get

$$\Delta_B[A] = \quad a(i_1,1)a(i_2, 2)\ldots.a(i_n, n)\ t(\ i_1, i_2, \ldots.i_n)\ \det\,[B].$$

i.e d et([B][A]) ={ $a(i_1,1)a(i_2, 2)\ldots.a(i_n, n)\ t(\ i_1, i_2, \ldots.i_n)$ }det [B], …(4)

for all n by n matrices [A] and [B].

Put B=I in (4).

Then,

$$\det\,([I][A]) = \{\quad a(i_1,1)a(i_2, 2)\ldots.a(i_n, n)\ t(\ i_1, i_2, \ldots.i_n)\ \}\det\,[I].$$

Since det [I] =1 and

$$[I][A] = [A], \text{ we get,}$$

$$\det\,[A] = \quad a(i_1,1)a(i_2, 2)\ldots.a(i_n, n)\ t(\ i_1, i_2, \ldots.i_n).$$

Substituting this in (4), we get,

$$\det([B][A]) = \det\,[A]\det\,[B].$$

Hence the theorem.

Theorem13.3

A linear operator A on R^n is invertible if and only if det [A] 0.

Proof

Let the linear operator A on R^n be invertible.

Then $[A][A^{-1}] = I.$

Therefore,

$$\det([A][A^{-1}]) = \det[I] = 1. \qquad \ldots(1)$$

By the theorem13.2,

$$\det([A][A^{-1}]) = \det[A]\det[A^{-1}]. \qquad \ldots(2)$$

From(1) and (2),

$$\det[A]\det[A^{-1}] = 1.$$

Hence det[A]   0.

Conversely assume that det [A]   0.
To prove that A is invertible.
Suppose if A is not invertible.
Let the column vectors of A be $\mathbf{x_1}, \mathbf{x_2}, \ldots \mathbf{x_n}$.
Since A is not invertible,
the column vectors are dependent.
Hence there is one $\mathbf{x_k}$ such that

$$\mathbf{x_k} + \sum_{j \neq k} c_j x_j = \mathbf{0}, \text{ for certain scalars } c_j.$$

By the properties (b) and (d) of theorem13.1,
$\mathbf{x_k}$ can be replaced by $\mathbf{x_k} + c_j\mathbf{x_j}$ without altering the determinant if j   k.
Repeating this, we can replace $\mathbf{x_k}$ by $\mathbf{x_k} + \sum_{j \neq k} c_j x_j$, i.e by 0,
without altering the determinant.
But determinant of a matrix which has **0** for one column is 0.
i.e det [A] = 0, which is a contradiction.
Hence A is invertible.
Hence the theorem.

Check your progress
Show that the determinant of the matrix of a linear operator does not depend on the basis which is used to construct the matrix.

Answer
Suppose { $\mathbf{e_1}, \mathbf{e_2}, \ldots \mathbf{e_n}$ }and { $\mathbf{u_1}, \mathbf{u_2}, \ldots \mathbf{u_n}$ }are bases in R^n.
Every linear operator A on R^n determines matrices [A] and $[A]_U$, with entries a_{ij} and α_{ij}.
The jth column of [A] is given by

$$A\mathbf{e_j} = \sum_i a_{ij} e_i \text{ , j=1, 2, \ldots n} \qquad \ldots(1)$$

and the jth column of $[A]_U$ is given by

$$A\mathbf{u_j} = \sum_k \alpha_{kj} u_k \qquad \text{, j=1, 2, \ldots n} \qquad \ldots(2)$$

If $\mathbf{u_k} = B\mathbf{e_k} =$ b $_{ik}e_i$, …(3)

then from (2),

$$\begin{aligned} A\mathbf{u_j} &= \sum_k \alpha_{kj} Be_k \\ &= \sum_k \alpha_{kj} \sum_i b_{ik} e_i \\ &= \sum_i \Big(\sum_k b_{ik}\alpha_{kj}\Big)e_i \qquad \ldots(4) \end{aligned}$$

From (3),

$$B\mathbf{e_j} = \ b_{kj}e_k$$

Therefore,

$$\begin{aligned} A\mathbf{u_j} &= Ab\mathbf{e_j} \\ &= \mathbf{A}\sum_k b_{kj}e_k \\ &= \sum_k b_{kj}Ae_k \\ &= \sum_k b_{kj}\ \Big(\sum_i a_{ik}e_i\ \Big) \\ &= \sum_i \Big(\sum_k a_{ik}b_{kj}\Big)e_i\ . \end{aligned} \qquad \ldots(5)$$

From (4) and (5),

$$\sum_k b_{ik}\alpha_{kj} \ = \ \sum_k a_{ik}b_{kj}$$

Hence $[B][A]_U = [A][B]$.

Therefore ,

$$\det([B][A]_U) \ = \det(\ [A][B])$$

Therefore,

$$\det[B]\det[A]_U = \det[A]\det[B] \qquad \ldots(6)$$

Since B is invertible,

$$\det[B] \quad 0.$$

Hence (6) implies ,

$$\det[A]_U = \det[A].$$

Hence the determinant of the matrix of a linear operator does not depend on the basis which is used to construct the matrix.

13.3 Jacobians

If **f** maps an open set $E \subset R^n$ into R^n , and if **f** is differentiable at a point $\mathbf{x} \in E$, the determinant of the linear operator **f'(x)** is called the Jacobian of **f** at **x**.

In Symbols,

$$\mathbf{J_f(x)} = \det \mathbf{f'(x)}.$$

If $(y_1, y_2, \ldots. Y_n) = \mathbf{f}(x_1, x_2, \ldots x_n)$,

then **J**$\mathbf{J_f(x)}$ is denoted by $\dfrac{\partial(y_1, y_2, \ldots. y_n)}{\partial(x_1, x_2, \ldots. x_n)}$.

13.4 Let us sum up

In this lesson we have seen

- the definition of determinant of the matrix of a linear operator
- Some of the properties of the determinant
- Determinant of the product of two matrices is the product of the determinants.
- The matrix of a linear operator does not depend on the basis used to construct the matrix and
- The Jacobian of a differentiable function.

13.5 Lesson End Activities

1. Set $f = (f_1, f_2, f_3)$ be the vector valued function defined for each pt: $(x_1, x_2, x_3) \in R^3$ (for which $x_1 + x_2 + x_3 \neq -1$) as follows:

$F_k(x_1, x_2, x_3) = x_k / 1 + x_1 + x_2 + x_3$ (k=1,2,3)
Show that $J_f(x_1, x_2, x_3) = (1 + x_1 + x_2 + x_3)^{-4}$
Show that f is $|-|$ and find f^{-1} explicitly

13.6 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

LESSON - 14

DERIVATIVES OF HIGHER ORDER

Contents

14.0 Introduction

In this lesson, we are going to study about the second order partial derivatives of a function.

14.1 Aims and objectives

After studying this lesson you would know

- The definition of second order partial derivatives of a function
- The mean value theorem for a function of two variables and
- The necessary conditions for D_{ij} **f** = D_{ji} **f**.

14.2 Derivatives of higher order

Definition

Suppose f is a real function defined in an open set $E \subset R^n$,
with partial derivatives $D_1 f, D_2 f, \ldots D_n f$.
If the functions $D_j f$ are themselves differentiable,
then the second –order partial derivatives of f are defined by

$$D_{ij} f = D_i D_j f \quad i, j=1, 2, \ldots n.$$

If all these functions $D_{ij} f$ are continuous in E,
then f is said to be of class C'' in E, or that $f \in C''(E)$.

A mapping **f** of E into R^m is said to be of class C'' **,** if each component of **f** is of class C''.

Theorem14.1 Mean value theorem for functions of two variables

Suppose f is defined in an open set $E \subset R^2$, and
D_1 f and D_{21} f exist at every point of E.
Suppose $Q \subset E$ is a closed rectangle with sides parallel to the coordinate axes,
having (a, b) and (a+h, b+k) as opposite vertices (h 0, k 0).
Put

$$\Delta(f, Q) = f(a+h, b+k) – f(a+h, b) – f(a, b+k) + f(a, b).$$

Then there is a point (x, y) in the interior of Q such that

$$\Delta(f, Q) = hk(D_{21}f)(x, y).$$

Proof

Let $u(t) = f(t, b+k) – f(t, b)$. …(1)

Then ,

$$\Delta(f, Q) = f(a+h, b+k) – f(a+h, b) – f(a, b+k) + f(a, b)$$
$$=u(a+h) – u(a)$$
$$=hu'(x) \quad \text{for some } x \in (a, a+h) \qquad \ldots(2)$$

[by the mean value theorem of real functions of single variables, "If f is a real continuous function on [a, b] which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which

$$f(b) - f(a) = (b-a)f'(x). \text{ "]}$$

From (1),

$$u'(t) = D_1f(t, b+k) - D_1f(t, b).$$

Substituting in (2),

$$\Delta(f, Q) = h[\, D_1f(x, b+k) - D_1f(x, b)]. \qquad \ldots(3)$$

Again applying the mean value theorem,

there exists a point $y \in (b, b+k)$ such that

$D_1f(x, b+k) - D_1f(x, b) = kD_{21}f(x, y)$

Substituting in(3), we get that

$$\Delta(f, Q) = hkD_{21}f(x, y).$$

Hence there is a point (x, y) in the interior of Q such that

$$\Delta(f, Q) = hk(D_{21}f)(x, y).$$

Hence the theorem

Theorem14.2

Suppose f is defined in an open set $E \subset R^2$,
suppose that D_1f, $D_{21}f$, and D_2f exist at every point of E, and $D_{21}f$ is continuous at some point $(a, b) \in E$.
Then $D_{12}f$ exists at (a, b) and

$$(D_{12}f)(a, b) = (D_{21}f)(a, b) .$$

Proof

Let $A = (D_{21}f)(a, b)$.

Choose $\varepsilon > 0$.

Let $Q \subset E$ be a closed rectangle with sides parallel to the coordinate axes, having (a, b) and (a+h, b+k) as opposite vertices (h 0, k 0).

Then by theorem14.1,

$$\Delta(f, Q) = f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)$$
$$= hk(D_{21}f)(x, y), \qquad \text{for some point } (x, y) \in Q. \qquad \ldots.(1)$$

If h and k are sufficiently small, then we have

$$|A - (D_{21}f)(x, y)| < \varepsilon \qquad \text{for all } (x, y) \in Q. \qquad \ldots.(2).$$

From (1) and (2), we get

$$|A - \frac{\Delta(f,Q)}{hk}| < \varepsilon.$$

i.e $$\left|\frac{\Delta(f,Q)}{hk} - A\right| < \varepsilon$$

i.e $$\left|\frac{f(a+h, b+k) - f(a+h, b) - f(a, b+k) + f(a, b)}{hk} - A\right| < \varepsilon$$

i.e $$\left|\frac{\dfrac{f(a+h,b+k) - f(a+h,k)}{k} - \dfrac{f(a,b+k) - f(a,b)}{k}}{h} - A\right| < \varepsilon$$

Keeping h fixed, let k › 0 .

Since D_2f exists in E, the last inequality implies that

$$\left|\frac{D_2 f(a+h,b) - D_2 f(a,b)}{h} - A\right| \quad \varepsilon \qquad \ldots(3)$$

Since ε was arbitrary, and
since (3) holds for all sufficiently small h 0,
(3) implies that $(D_{21}f)(a, b)$ exists and

$$(D_{21}f)(a, b) = A.$$

Hence $D_{12}f)(a, b) = (D_{21}f)(a, b)$.

Check your progress.

Put f(0, 0) = 0. and

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2} \qquad \text{if } (x, y)\ (0, 0).$$

Prove that (a) f, $D_1 f$, $D_2 f$ are continuous in R^2.
(b) $D_{12}f$ and $D_{21}f$ exist at every point of R^2, and are continuous except at (0, 0)
(c)($D_{12}f)(0, 0) = 1$ and $(D_{21}f)(0, 0) = -1$.

14.3Let us sum up

In this lesson we have seen

- The definition of second order partial derivatives which we can extend to higher order partial derivatives.
- Mean value theorem for functions of two variables. and
- Conditions to be satisfied in order that the partial derivatives $D_{12}f$ and $D_{21}f$ are equal.

14.4 Lesson End Activities

1. if $f(x,y) = xy/ x^2+y^2$ if $(x,y) \neq (0,0) = 0$ if$(x,y) = (10,0)$. Prove that both the partial dervivaties exist at (0,0) but the function is not continous at (0,0))
2. Expand $x^3y+3y-z$ in powers of (x-1) and (y+2).

14.5 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

LESSON - 15

DIFFERENTIATION OF INTEGRALS

15.0 Introduction
15.1 Aims and objectives
15.2 Differentiation of Integrals
15.3 Let us sum up
15.4 References

15.0 Introduction

In this lesson, we are going to study about differentiation of integrals. Suppose φ is a function of two variables which can be integrated with respect to one variable and which can be differentiated with respect to another variable. We are going to see under what conditions the result will be the same if these two limit process are carried out in the opposite order.

15.1 Aims and objectives

After studying this lesson, you would know

- Under what conditions the following equation is true?.

$$\frac{d}{dt}\int_a^b \varphi(x,t)dx = \int_a^b \frac{\partial\varphi}{\partial t}(x,t)dx$$

Note:

It will be convenient to use the notation

$$\varphi^t(x) = \varphi(x, t)$$

Thus φ^t is a function of one variable, for each t.

15.2 Differentiation of Integrals

Theorem15.1

Suppose

(a) $\varphi(x, t)$ is defined for $a \quad x \quad b,\ c \ t \ b$;

(b) α is an increasing function on [a, b];

(c) $\varphi^t \in \mathcal{R}(\alpha)$ for every $t \in [c, d]$;

(d) c<s<d, and to every ε >0 corresponds a δ > 0 such that

$$|(D_2\,\varphi)(x, t) - (D_2\,\varphi)(x, s)| < \varepsilon$$

for all $x \in [a, b]$ and for all $t \in (s - \delta , s + \delta)$.

Define

$$f(t) = \int_a^b \varphi(x,t)d\alpha(x) \quad c \quad t \quad d.$$

Then $(D_2\,\varphi)^s \in \mathcal{R}(\alpha)$,

f'(s) exists, and

$$f'(s) = \int_a^b (D_2\varphi)(x,s)d\alpha(x).$$

Proof

For $0 < |t - s| < \delta$,

consider the difference quotients

$$\psi(x, t) = \frac{\varphi(x,t) - \varphi(x,s)}{t - s}.$$

Since $\varphi^t \in \mathcal{R}(\alpha)$ for every $t \in [c, d]$,

$$\psi^t \in R(\alpha), \text{ for each } t \in [c, d].$$

By mean value theorem,

there exists a u between s and t such that

$$\varphi(x, t) - \varphi(x, s) = D_2\, \varphi(x, u)(t - s)$$

Hence to each (x, t),

there exists a u between s and t such that

$$\psi(x, t) = D_2\, \varphi(x, u) \quad \text{whenever } 0 < |t - s| < \delta. \qquad \ldots(1)$$

Hence assumption (d) implies that

$$|\psi(x, t) - (D_2\, \varphi)(x, s)| < \varepsilon \qquad \ldots(2)$$

whenever a x b and 0 < |t – s | <δ.

Consider

$$\frac{f(t) - f(s)}{t - s} = \frac{1}{t - s}\left[\int_a^b \varphi(x,t)\,d\alpha(x) - \int_a^b \varphi(x,s)\,d\alpha(x)\right]$$

$$= \int_a^b \frac{\varphi(x,t) - \varphi(x,s)}{t - s}\,d\alpha(x)$$

$$= \int_a^b \psi(x,t)\,d\alpha(x) \qquad \ldots(3)$$

From(2), we get that

$$\psi^t \,›\, (D_2\varphi)^s, \text{ uniformly on [a. b], as } t \,›\, s. \qquad \ldots(4)$$

Also each $\psi^t \in R(\alpha)$.

Hence $\quad f'(s) = \lim_{t › s} \dfrac{f(t) - f(s)}{t - s}$

$$= \lim_{t › s} \int_a^b \psi(x,t)\,d\alpha(x)$$

$$= \int_a^b \lim_{t \to s} \psi(x,t)\,d\alpha(x)$$

$$= \int_a^b (D_2\varphi)(x,s)\,d\alpha(x) \qquad \text{[from (4)and theorem 7.1]}$$

Hence the theorem.

Example

Define $\quad f(t) = \int_{-\infty}^{\infty} e^{-x^2} \cos(xt)\,dx$

And $\quad g(t) = -\int_{-\infty}^{\infty} xe^{-x^2} \sin(xt)\,dx$, for - < t < .

Both integrals exist since the absolute values of the integrands are Atmost exp(-x^2) and |x| exp(-x^2), respectively.

Also g can be obtained from f by differentiating the integrand of f(t) with respect to t.

Claim: f ‘(t) = g(t) - < t < .

Proof

By integrating the right hand side, we can prove that

$$\frac{\cos(\alpha + \beta) - \cos\alpha}{\beta} + \sin\alpha = \frac{1}{\beta}\int_\alpha^{\alpha+\beta} (\sin\alpha - \sin t)\,dt \text{ , for } \beta > 0.$$

Therefore

$$\left|\frac{\cos(\alpha + \beta) - \cos\alpha}{\beta} + \sin\alpha\right| = \left|\frac{1}{\beta}\int_\alpha^{\alpha+\beta} (\sin\alpha - \sin t)\,dt\right|$$

$$\frac{1}{\beta}\int_\alpha^{\alpha+\beta} |(\sin\alpha - \sin t)|\,dt$$

$$\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} |\alpha - t|\, dt$$

$$\frac{1}{\beta}\int_{\alpha}^{\alpha+\beta} (t-\alpha)dt$$

$$\frac{1}{\beta}\left[\frac{(t-\alpha)^2}{2}\right]_{\alpha}^{\alpha+\beta}$$

β/2
β

Similar result can be obtained if β < 0.
Therefore

$$\left|\frac{\cos(\alpha+\beta)-\cos\alpha}{\beta}+\sin\alpha\right| \qquad |\beta| \text{ for all } \beta.$$

Fix t, and fix h> 0.
Substituting α =xt, β=xh in the above inequality, we get

$$\left|\frac{\cos(t+h)x-\cos tx}{xh}+\sin tx\right| \qquad |x|\ |h|$$

Therefore,

$$\left|\frac{\cos(t+h)x-\cos tx}{h}+x\sin tx\right| \quad |x|^{2}\ |h|. \qquad \ldots(1).$$

Consider

$$\left|\frac{f(t+h)-f(t)}{h}-g(t)\right|$$

$$=\left|\frac{\int_{-\infty}^{\infty} e^{-x^2}\cos(t+h)x\,dx-\int_{-\infty}^{\infty} e^{-x^2}\cos(tx)dx}{h}+\int_{-\infty}^{\infty} xe^{-x^2}\sin(xt)dx\right|$$

$$=\left|\int_{-\infty}^{\infty} e^{-x^2}\left(\frac{\cos(t+h)x-\cos(tx)}{h}+x\sin(tx)\right)dx\right|$$

$$\int_{-\infty}^{\infty} e^{-x^2}\left|\left(\frac{\cos(t+h)x-\cos(tx)}{h}+x\sin(tx)\right)\right|dx$$

$$\int_{-\infty}^{\infty} e^{-x^2}\,|x|^2|h|\,dx$$

$|h| \int_{-\infty}^{\infty} e^{-x^2}x^2dx$ › 0 as h › 0.

Hence f ‘(t) = g(t) - < t < .
Hence the required result.

15.3 Let us sum up

In this lesson, we have seen

- under what conditions a function of two variables can be integrated with respect to one variable and can be differentiated with respect to another variable and
- an example .

15.4 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

Unit 4 Lebesgue Measure and Lebesgue integral.

Lesson 16 Outer measure, Measurable sets and Lebesgue Measure

16.0 Introduction

In this lesson, we are going to study about Lebesgue outer measure of a set, measurable sets and Lebesgue measure, their important properties and littlewood's First principle.

16.1 Aims and Objectives

After studying this lesson , you would know

- The definition of outer measure of sets
- Outer measure of an interval
- Some important properties of Outer measure
- The definition of Measurable sets
- Measure of countable union of measurable sets
- Measure of Countable intersection of measurable sets and
- Littlewood's First Principle.

16.2 Outer measure

Definition

A collection $\mathcal{M}$ of sets is called a σ – algebra or a Borel set if

(i) Every union of a countable collection of sets in $\mathcal{M}$ is again in $\mathcal{M}$ and

(ii) If $A \in \mathcal{M}$, then $\tilde{A} \in \mathcal{M}$.

Definition : Outer measure

Let A be set of real numbers.

The outer measure $m^*(A)$ of A is defined as

$$m^*(A) = \inf_{A \subset \bigcup I_n} \sum l(I_n)$$

where $l(I_n)$ denotes the length of I_n.

Theorem16.1

(a) If $A \subset (a, b)$, then $m^*(A) \quad (b – a)$.
(b) If $A \subset B$, then $m^*(A) \quad m^*(B)$
(c) $m^*(\{x\}) =0$, where x is a real number.
(d) $m^*(\phi) =0$

Proof

(a) Let $U = \{\bigcup I_n \;/\; A \subset \bigcup I_n\}$

Since A $\subset$ (a, b),

(a, b) $\in$ U.

Therefore,

$$\inf_{A\subset\cup I_n} \sum l(I_n) \quad \text{l(a, b)}$$

$$=(b – a)$$

Therefore,

m*(A)   (b – a).

(b) Let A $\subset$ B.

Let U = {$\bigcup$I$_n$ / A$\subset\bigcup$I$_n$} and

V ={ $\bigcup$J$_n$ / B$\subset\bigcup$J$_n$

Since A $\subset$B,

A $\subset\bigcup$J$_n$.

Therefore every element of V is an element of U .

Hence V $\subset$ U.

Hence ,

$$\inf_{A\subset\cup I_n} \sum l(I_n) \qquad \inf_{B\subset\cup J_n} \sum l(J_n)$$

Hence m*(A)   m*(B).

(c) Let   x $\in$ R.

For any ε >0,

{x} $\subset$(x- ε, x+ ε) and

l(x- ε, x+ ε) =2 ε.

Therefore by (a),

m*({x})   2 ε.

Since ε was arbitrary,

this implies   m*({x}) = 0.

(e) Let x $\in$R.

Then ϕ $\subset${x}

Therefore by (b),

m*(ϕ) m*({x})

Since m*({x}) = 0,

m*(ϕ) = 0.

Hence the theorem.

Theorem16.2

The outer measure of an interval is its length.

Proof

Case 1. Let I be a finite closed interval, say [a, b].

Therefore,

I $\subset$ (a – ε, a + ε) for any ε > 0.

Therefore,

m*(I)   b – a +2 ε.   [by the theorem16.1(a)]

Since this is true for every ε > 0,

m*(I)   b – a   … (1)

Let $\quad I \subset \bigcup_{n=1}^{\infty} I_n$.

By Heine –Borel theorem,

There exists a finite subcollection from { I_n }, say $I_1, I_2, \ldots I_m$, which covers I, and

$$\sum_{k=1}^{m} l(I_k) \quad \sum_{n=1}^{\infty} l(I_n).$$

Since $\quad I=[a, b] \subset I_1 \cup I_2 \cup \ldots \cup I_m$,

$a \in I_l$, for some l with 1 l m.

Let $\quad I_l$ be (a_1, b_1).

Therefore $\quad a_1 < a < b_1$.

If $\ b < b_1$,

then $a_1 < a < b < b_1$. i.e $(a, b) \subset I_l$.

Therefore $\quad l(I_l) \quad b - a$.

Therefore $\quad \sum_{n=1}^{\infty} l(I_n) \quad l(I_l) \quad b - a$.

If $\ b_1 \quad b$,

then $b_1 \in [a, b]$ but $\ b_1 \notin I_l$.

Therefore $\ b_1 \in I_t$ for some t, 1 t m.

Let $\quad I_t$ be (a_2, b_2).

Therefore $\quad a_2 < b_1 < b_2$.

If $\ b < b_2$,

then $a_1 < a < b_1 \quad b < b_2$.

Therefore,

$$\sum_{n=1}^{\infty} l(I_n) \quad l(I_l) + l(I_t)$$

$=(b_2 - a_2) + (b_1 - a_1)$

$=(b_2 - a_1) + (b_1 - a_2)$

$> (b_2 - a_1)$

$\quad b - a$.

If $\ b_2 \quad b$,

continuing in the same way, we obtain a sequence

$(a_1, b_1), (a_2, b_2), \ldots (a_k, b_k)$ from $\{I_n\}_{n=1}^{m}$ such that $a_i < b_{i-1} < b_i$.

Since $\{I_n\}_{n=1}^{m}$ is a finite collection, this process must terminate with some interval say (a_r, b_r).

But it terminates only if $b \in (a_r, b_r)$.

Therefore ,

$$\sum_{n=1}^{\infty} l(I_n) \quad \sum_{i=1}^{r} l(a_i, b_i)$$

$=(b_r - a_r) + (b_{r-1} - a_{r-1}) + \ldots + (b_1 - a_1)$

$= b_r - (a_r - b_{r-1}) - \ldots - (a_2 - b_1) - a_1$.

$\quad b_r - a_1$. [since $a_i < b_{i-1}$, 1 i r.

$\quad b - a$.

Therefore, $\quad \inf \sum_{n=1}^{\infty} l(I_n) \quad b - a$.

Hence $\quad m^*(I) \quad b - a \quad \ldots(2)$

From (1) and (2),

$$m^*(I) = b - a.$$

Case 2: Let I be any finite interval
i.e I =[a, b), or (a, b), or (a, b].

Then there exists an interval $J = [a+\varepsilon/4,\ b - \varepsilon/4]$ such that $J \subset I$.

$$\begin{aligned} \text{But} \qquad m^*(J) &= l(J) \\ &= (b - \varepsilon/4) - (a + \varepsilon/4) \qquad [\text{ by case } 1] \\ &= b - a - \varepsilon/2 \\ &= l(I) - \varepsilon/2 \\ &> l(I) - \varepsilon \end{aligned}$$

Then for each $\varepsilon > 0$,

$$\begin{aligned} l(I) - \varepsilon &< l(J) \\ &= m^*(J) \\ &\quad m^*(I) \\ &\quad m^*(\bar{I}) \\ &= l(\bar{I}) \\ &= l(I). \end{aligned}$$

Therefore

$$l(I) - \varepsilon < m^*(I) \quad l(I).$$

Since this is true for every $\varepsilon > 0$, we get that

$$m^*(I) = l(I).$$

case 3. Let I be any infinite interval.
Then given any real number Δ,
there is a closed interval $J \subset I$ with $l(J) = \Delta$.
Hence ,

$$m^*(I) \quad m^*(J) = l(J) = \Delta.$$

Therefore ,

$$m^*(I) \quad \Delta \text{ for each } \Delta > 0.$$

This implies ,

$$m^*(I) = \quad = l(I).$$

Hence in all cases $m^*(I) = l(I)$.
Hence the theorem.

Theorem16.3

Let $\{A_n\}_{n=1}^{\infty}$ be a countable collection of sets of real numbers.

Then $$m^*\Big(\bigcup_{n=1}^{\infty} A_n\Big) \quad \sum_{n=1}^{\infty} m^*(A_n).$$

Proof

Suppose $m^*(A_n) =$ for some n.

Then $$\sum_{n=1}^{\infty} m^*(A_n) = \quad .$$

Hence $$m^*\Big(\bigcup_{n=1}^{\infty} A_n\Big) \quad \sum_{n=1}^{\infty} m^*(A_n).$$

Hence the theorem is true.

So assume that $m^*(A_n) <$, for all n.

i.e inf $\inf_{A_n \subset \bigcup I_{n,i}} \sum l(I_{n,i}) <$, for all n.

Hence for given ε > 0,

there exist countable collection $\{ I_{n,i}\}_i$ of open intervals such that

$$A_n \subset \bigcup_i I_{n,i} \text{ and}$$

$$\sum_i l(I_{n,i}) < m^*(A_n) + 2^{-n} \varepsilon \qquad \ldots(1)$$

Since a union of a countable number of countable collections is countable,

$\{ I_{n,i}\}_{n,i}$ is countable and

$$\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n,i} I_{n,i}$$

Hence
$$m^*(\bigcup_i I_{n,i}) \quad \sum_{n,i} l(I_{n,i})$$

$$= \sum_n \sum_i l(I_{n,i})$$

$$< \sum_n (m^* A_n + 2^{-n} \varepsilon)$$

$$< \sum_n (m^* A_n) + \sum_n (2^{-n} \varepsilon)$$

$$= \sum_n (m^* A_n) + \varepsilon$$

Since ε was an arbitrary positive number,

$$m^*(\bigcup_{n=1}^{\infty} A_n) \quad \sum_{n=1}^{\infty} m^*(A_n).$$

Hence the theorem.

Remark:

- This property is called countable subadditivity of m*.
- Clearly m* satisfies the finite subadditivity

  i.e $m^*(\bigcup_{m=1}^{n} A_m) \quad \sum_{m=1}^{n} m^*(A_m).$

Theorem16.4

If A is countable, prove that m*(A) = 0.

Proof

Let $A=\{ x_1 , x_2 , \ldots.. x_n , \ldots..\}$.

$$m^*(A) = m^* \left[\bigcup_i \{x_i\} \right]$$

$$\sum_i m^*(\{x_i\}) \qquad \text{[by theorem16.3]}$$

$$=0 \qquad \text{[since } m^*(\{x\}) = 0, \text{ for all real x]}$$

Hence m*(A) = 0.

Hence the theorem.

Theorem16.5

The set [0, 1] is not countable.

Proof

On the contrary ,

assume that [0, 1] is countable.

Then by theorem 16.4,

$$m^*([0, 1]) = 0$$

But by theorem16.2,

$$m^*([0, 1]) =(1 – 0) =1$$

which is a contradiction.

Hence [0, 1] is not countable.

Check your progress

1. If Q denotes the set of all rational numbers, prove that m*(Q) = 0

Given any set A and $\varepsilon > o$, Prove that

there is an open subset U such that $A \subset U$ and

$m^*(U) = m^*(A) + \varepsilon$.

2. m* is translation invariant i.e $m^*(A+x) = m^*(A)$, for all $x \in R$

3. Prove that if m*(A)= 0, then m*(AUB)=m*(B).

4. Let A be the set of all rational numbers between 0 and 1.

Let $\{I_n\}$ be a finite collection of open intervals covering A.

Then $l(I_n)$ 1 .

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16.3 Measurable sets

Definition

A subset E of R is said to be measurable,

if for each subset A of R, we have

$$m^*A = m^*(A \cap E) + m^*(A \cap \tilde{E})$$

where $\tilde{E} = R – E$.i.e $\tilde{E}$ denotes the complement of E in R.

Remark:

1. Since we are using Lebesgue outer measure m*, the measurable sets are called Lebesgue measurable sets.
2. Since $A = (A \cap E) \cup (A \cap \tilde{E})$,

   $m^*A \quad m^*(A \cap E) + m^*(A \cap \tilde{E})$

   Therefore to show that E is measurable,

   it is enough to show that

   $m^*A \quad m^*(A \cap E) + m^*(A \cap \tilde{E})$.

3.Since the definition of measurability is symmetric in E and $\tilde{E}$,
$\tilde{E}$ is measurable whenever E is measurable.

Theorem16.6

φ and R are measurable sets.

Proof

If E = R,

then $A \cap R = A$ and

$$A \cap \tilde{R} = A \cap \varphi$$
$$= \varphi.$$

Therefore,

$$m^*(A \cap R) = m^*A \text{ and}$$

$$m^*(A \cap \tilde{R}) = m^* \varphi = 0.$$

Therefore,

$$m^*A = m^*(A \cap R)$$
$$= m^*(A \cap R) + m^*(A \cap \tilde{R})$$

Hence R is measurable.
Therefore ,

$\tilde{R}$ is also measurable.
i.e φ is measurable.

Hence R and φ are measurable sets.
Hence the theorem.

Theoem16.7

If $m^*E = 0$, then E is measurable.

Proof

Let A be any subset of R.

Then $A \cap E \subset E.$

Therefore,

$$m^*(A \cap E) \quad m^*E = 0.$$

Therefore,

$$m^*(A \cap E) = 0. \qquad \ldots(1)$$

But $A \cap \tilde{E} \subset A.$

Therefore,

$$m^*A \quad m^*(A \cap \tilde{E}) \qquad \ldots(2)$$

From (1) and (2), we get

$$m^*A \quad m^*(A \cap E) + m^*(A \cap \tilde{E}).$$

Hence E is measurable.
Hence the theorem.

Theorem16.8

If E_1 and E_2 are measurable sets , then $E_1 \cup E_2$ is also measurable.

Proof

Let A be any subset of R.
Since E_2 is measurable,

$$m^*A = m^*(A \cap E_2) + m^*(A \cap \tilde{E}_2).$$

Replacing A by $A \cap \tilde{E}_1$,

$$m^*(A \cap \tilde{E}_1) = m^*(A \cap \tilde{E}_1 \cap E_2) + m^*(A \cap \tilde{E}_1 \cap \tilde{E}_2). \qquad \ldots(1)$$

Consider,

$$A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2)$$
$$= (A \cap E_1) \cup (A \cap E_2 \cap R)$$
$$= (A \cap E_1) \cup (A \cap E_2 \cap (\tilde{E}_1 \cup E_1))$$
$$= (A \cap E_1) \cup [(A \cap E_2 \cap \tilde{E}_1) \cup (A \cap E_2 \cap E_1)]$$
$$= (A \cap E_1) \cup (A \cap E_2 \cap \tilde{E}_1)$$

[since $A \cap E_2 \cap E_1 \subset A \cap E_1$]

Therefore,

$$m^*(A \cap (E_1 \cup E_2)) \quad m^*(A \cap E_1) + m^*(A \cap E_2 \cap \tilde{E}_1)$$

Adding $m^*(A \cap \tilde{E}_1 \cap \tilde{E}_2)$ to both sides,

$$m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \tilde{E}_1 \cap \tilde{E}_2)$$

$$m^*(A \cap E_1) + m^*(A \cap E_2 \cap \tilde{E}_1) + m^*(A \cap \tilde{E}_1 \cap \tilde{E}_2)$$

$$= m^*(A \cap E_1) + m^*(A \cap \tilde{E}_1) \qquad \text{[using (1)]}$$

$$= m^*A \qquad \text{[Since } E_1 \text{ is measurable.]}$$

Therefore ,

$$m^*A \quad m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \tilde{E}_1 \cap \tilde{E}_2)$$

$$m^*A \quad m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (\tilde{E}_1 \cup \tilde{E}_2))$$

Hence $E_1 \cup E_2$ is measurable.

Hence the theorem.

Check your progress

1. Prove that the family $\mathcal{M}$ of measurable sets is an algebra.
2. If $E_1, E_2, \ldots E_n$ are measurable, prove that $E_1 \cup E_2 \cup \ldots \cup E_n$ is measurable.
3. If E_1 and E_2 are measurable sets , then prove that $E_1 \cup E_2$ is also measurable.

Theorem16.9

Let A be any set and $E_1, E_2, \ldots E_n$ a finite sequence of disjoint measurable sets,

Then $m^*(A \cap \left[\bigcup_{i=1}^{n} E_i\right]) = \sum_{i=1}^{n} m^*(A \cap E_i)$.

Proof

We prove this theorem , by induction on n.

When n=1, the result is obvious.

Assume that the result is true for n-1 sets(n 2).

i.e $m^*(A \cap \left[\bigcup_{i=1}^{n-1} E_i\right]) = \sum_{i=1}^{n-1} m^*(A \cap E_i)$. …(1)

Let $E_1, E_2, \ldots E_n$ be n disjoint measurable sets.

Then $A \cap \left[\bigcup_{i=1}^{n} E_i\right] \cap E_n = A \cap E_n$ …(2)

$$\left[\text{since } E_n \subset \left[\bigcup_{i=1}^{n} E_i\right]\right]$$

$$A \cap \left[\bigcup_{i=1}^{n} E_i\right] \cap \tilde{E}_n = A \cap [(E_1 \cap \tilde{E}_n) \cup \ldots \cup (E_n \cap \tilde{E}_n)$$

Since $E_1, E_2, \ldots E_n$ are disjoint ,

$E_i \subset E_n$ for i n, and

$E_n \cap \tilde{E}_n = \varphi$, we get

$$A \cap \left[\bigcup_{i=1}^{n} E_i\right] \cap \tilde{E}_n = A \cap [E_1 \cup E_2 \cup \ldots \cup E_{n-1} \cup \varphi]$$

$$= A \cap \left[\bigcup_{i=1}^{n-1} E_i\right] \qquad \ldots(3)$$

Since E_n is measurable,

$$m^*A = m^*(A \cap E_n) + m^*(A \cap \tilde{E}_n).$$

Replacing A by $A \cap \left[\bigcup_{i=1}^{n} E_i\right]$, we get,

$$m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right] \cap E_n\right) + m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right] \cap \tilde{E}_n\right).$$

$$= m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_{i=1}^{n-1} E_i\right]\right) \qquad \text{[from (2) and (3)]}$$

$$= m^*(A \cap E_n) + \sum_{i=1}^{n-1} m^*(A \cap E_i) \qquad \text{[from(1)]}$$

Hence $m^*\left(A \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$.

Hence by induction the theorem follows.

Remark:

m* is finitely additive.

Proof:

In the above theorem, take A= R.

Then,

$$m^*\left(R \cap \left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^*(R \cap E_i).$$

Hence,

$$m^*\left(\left[\bigcup_{i=1}^{n} E_i\right]\right) = \sum_{i=1}^{n} m^* E_i .$$

Hence the result.

Defintion

A collection T of subsets of a set X is called a σ- algebra if

(i) $A \in T$ implies $\tilde{A} \in T$.

(ii) If $A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in T$, for every n, then $A \in T$.

Theorem16.10

The collection $\mathcal{M}$ of measurable sets is a σ- algebra.

Proof

If E is a measurable set,

then we know that $\tilde{E}$ is also measurable.

Hence if $E \in \mathcal{M}$, then $\tilde{E} \in \mathcal{M}$.

Hence the first condition of the definition of σ- algebra is satisfied.

To prove the second condition:

Let $E_n \in \mathcal{M}$, n=1,2,3,….

Let $E = \bigcup_{n=1}^{\infty} E_n$.

Claim: There is a set $F_n \in M$, n=1,2,3,… such that $F_i \cap F_j = \phi$ for i j and

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i \, .$$

To prove the claim,

Take $F_1 = E_1$.

For n> 1,

define $F_n = E_n - \{ E_1 \cup E_2 \cup \ldots \cup E_{n-1} \}$

$= E_n \cap \tilde{E}_1 \cap \ldots\ldots \cap \tilde{E}_{n-1.}$

Since M is an algebra,

$E_i \in M$ implies $\tilde{E}_i \in M$ 1 i n-1.

This implies,

$E_n \cap \tilde{E}_1 \cap \ldots\ldots \cap \tilde{E}_{n-1} \in M$.

This implies,

$F_n \in M$, n=1,2,3,….

Also $F_n \subset E_n$, for all n.

Let m n.

Suppose m<n.

Then ,

$F_m \cap F_n \subset E_m \cap F_n = E_m \cap (E_n \cap \tilde{E}_1 \cap \ldots\ldots \cap \tilde{E}_{n-1})$

$= E_n \cap \tilde{E}_1 \cap \ldots E_m \cap \tilde{E}_m \,..\cap \tilde{E}_{n-1}$

$= \phi$.

Hence if m n,

then $F_m \cap F_n = \phi$.

Since $F_i \subset E_i$, for all i,

$$\bigcup_{i=1}^{\infty} F_i \subset \bigcup_{i=1}^{\infty} E_i \, . \qquad \ldots(1)$$

Now, let $x \in \bigcup_{i=1}^{\infty} E_i$.

This implies,

$x \in E_n$ for some n.

Let m be the smallest value of n such that $x \in E_m$.

Then $x \notin E_i$, for i < m.

This implies that,

$x \in \tilde{E}_i$, for i <m.

Therefore,

$x \in E_m \cap \tilde{E}_{m-1} \cap \ldots\ldots \cap \tilde{E}_1$.

Therefore,

$$x \in F_m \subset \bigcup_{i=1}^{\infty} F_i \, .$$

Hence
$$\bigcup_{i=1}^{\infty} E_i \subset \bigcup_{i=1}^{\infty} F_i \qquad \ldots(2)$$

From(1) and (2), we get that

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i \ .$$

Hence the claim is proved.

i.e if E is the union of a countable collection of measurable sets, it must be the union of pairwise disjoint measurable sets.

Let $A_n = \bigcup_{i=1}^{n} E_i$.

Then $A_n \in \mathcal{M}$, [Since $\mathcal{M}$ is an algebra and $E_i \in \mathcal{M}$]

Hence A_n is measurable.

Also $A_n \subset E$.

Hence $\tilde{A}_n \supset \tilde{E}$

Therefore,

$A \cap \tilde{A}_n \supset A \cap \tilde{E}$, for any set A in R.

$$m^*(A \cap \tilde{A}_n)\ m^*(A \cap \tilde{E}) \qquad \ldots(3)$$

Since A_n is measurable,

$$m^*A = m^*(A \cap A_n) + m^*(A \cap \tilde{A}_n)$$

$$m^*(A \cap A_n) + m^*(A \cap \tilde{E})$$

$$= m^*(A \cap \bigcup_{i=1}^{n} E_i) + m^*(A \cap \tilde{E})$$

$$= \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap \tilde{E}) \text{ [by theorem 16.9]}$$

Since this is true for every n,

$$m^*A \qquad \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap \tilde{E})$$

$$m^*(A \cap E) + m^*(A \cap \tilde{E})$$

[since by theorem16.3 , $m^*(A \cap E) \quad \sum_{i=1}^{\infty} m^*(A \cap E_i)$].

Hence E is measurable.

i.e $E \in \mathcal{M}$.

Hence $\mathcal{M}$ is a σ – algebra.

Hence the theorem.

Remark:

The intersection of a countable collection of measurable sets is measurable.

Proof:

Since the complement of a measurable set is measurable,

$$(\bigcup_{i=1}^{\infty} E_i)^{\sim} \quad \text{is measurable.}$$

Hence ,

$(\bigcup_{i=1}^{\infty}\tilde{E}_i)^{\sim}$ is measurable.

i.e $\bigcap_{i=1}^{\infty}E_i$ is measurable.

Hence the required result.

Theorem16.11

The interval (a,) is measurable.

Proof

Let A be any subset of R.

Let $A_1 = A \cap (a, \ )$.

$A_2 = A \cap (a, \ )^{\sim}$

$= A \cap (- \ , \ a]$.

Therefore to show that (a,) is measurable,
we have to prove that

$m^*A \quad m^*A_1 + m^*A_2$.

If $m^*A = \ $, then there is nothing to prove.

Hence let $m^*A < \ $.

By definition,

$$m^*A = \inf_{A \subset \bigcup I_n} \sum l(I_n) \ .$$

Hence for given $\varepsilon > 0$,

there exist open intervals $\{I_n\}_{n=1}^{\infty}$ such that $A \subset \bigcup_{n=1}^{\infty} I_n$ and

$$\sum_{n=1}^{\infty} l(I_n) \quad m^*A + \varepsilon \ . \qquad \ldots(1)$$

Let $I_n' = I_n \cap (a, \ )$.

And $I_n'' = I_n \cap (- \ , \ a]$.

Then I_n' and I_n'' are intervals (or may be empty)

And $l(I_n) = l(I_n') + l(I_n'')$

$= m^*I_n' + m^*I_n''$. …(2)

Now $A_1 = A \cap (a, \ )$

$\subset (\bigcup_{n=1}^{\infty} I_n) \cap (a, \ )$

$= \bigcup_{n=1}^{\infty} I_n \cap (a, \ )$

$= \bigcup_{n=1}^{\infty} I_n'$.

i.e $A_1 \subset \bigcup_{n=1}^{\infty} I_n'$.

Therefore ,

$$m^*A_1 \subset m^*(\bigcup_{n=1}^{\infty} I_n') \quad \sum_{n=1}^{\infty} m^* I_n'$$

Similarly,

$$m^*A_2 \subset m^*(\bigcup_{n=1}^{\infty} I_n{''}) \quad \sum_{n=1}^{\infty} m^* I_n'' .$$

Therefore,

$$m^*A_1 + m^*A_2 \quad \sum_{n=1}^{\infty} m^* I_n' + \sum_{n=1}^{\infty} m^* I_n''$$

$$= \sum_{n=1}^{\infty} (m^* I_n' + m^* I_n'')$$

$$= \sum_{n=1}^{\infty} l(I_n) \qquad \text{[From (2)]}$$

$$m^*A + \varepsilon . \qquad \text{[From(1)].}$$

Since ε was arbitrary,

$$m^*A_1 + m^*A_2 \quad m^*A.$$

Hence (a,) is measurable.
Hence the theorem.

Definition: Borel set

The smallest σ – algebra which contains all open sets of R is called the collection of all Borel sets.

Theorem 16.12

Every Borel set is measurable.

Proof

To show that every Borel set is measurable,
it is enough to show that M contains all open sets.
i.e every open set of R is measurable.
Then M being a σ-algebra and
P being the smallest – algebra containing the open sets of R,

$$P \subset M .$$

And hence every element of P is measurable
i.e every Borl set is measurable.

Claim: Every open set of R is measurable.
Consider (a,), for any a ∈ R.
By theorem 16.11,
(a,) is measurable.
Therefore,
$(a,\ )^{\sim} = (-\ , a]$ is measurable. …(1)
For any b ∈ R,

$$(-\ , b) = \bigcup_{n=1}^{\infty} (-\infty, b - \frac{1}{n}].$$

By (1),
each $(-\ , b-\frac{1}{n}]$ is measurable.
Also union of countable number of measurable sets is measurable.

Hence (- , b) is measurable.

Now consider any open interval (a, b).
(a, b) can be written as

(a, b) = (- , b) $\cap$ (a,) .

As we have already seen ,

(- , b) and (a,) are measurable and
intersection of two measurable sets is measurable.

Hence (a, b) is measurable.
Since each open set of R is a countable union of open intervals,
each open set of R is measurable.
Hence our claim is proved.
Hence the theorem.

16.4 Lebesgue measure

Definition

The Lebesgue measure m is the set function from the family $\mathcal{M}$ of Lebesgue measurable sets to R $\cup$ { } defined by

m(E) = m*(E), for every E $\in$ $\mathcal{M}$.

i.e m: $\mathcal{M}$ › R $\cup$ { } defined by m(E) = m*(E), for every E $\in$ $\mathcal{M}$.

Theorem16.13

Let { E_i } be a sequence of pairwise disjoint measurable sets.

Then $m(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$.

Proof

Let E_1 , E_2 , …E_n be a finite sequence of measurable sets.
Then by theorem 16.9,

$$m^*(A \cap \left[\bigcup_{i=1}^{n} E_i\right]) = \sum_{i=1}^{n} m^*(A \cap E_i) . \qquad ...(1)$$

where A is any subset of R.
Taking A=R in (1), we get

$$m^*(R \cap \left[\bigcup_{i=1}^{n} E_i\right]) = \sum_{i=1}^{n} m^*(R \cap E_i) .$$

Hence,

$$m^*(\left[\bigcup_{i=1}^{n} E_i\right]) = \sum_{i=1}^{n} m^* E_i .$$

Hence ,

$$m(\bigcup_{i=1}^{n} E_i) = \sum_{i=1}^{n} mE_i . \qquad …(2)$$

Thus m is finitely additive..

Let { E_i} be an infinite sequence of positive disjoint measurable sets.
Then,

$$\bigcup_{i=1}^{\infty} E_i \supset \bigcup_{i=1}^{n} E_i \text{ , for any n.}$$

Therefore,

$$m^*\Big(\bigcup_{i=1}^{\infty} E_i\Big) \quad m^*\Big(\bigcup_{i=1}^{n} E_i\Big), \text{ for any n.}$$

Therefore,

$$m\Big(\bigcup_{i=1}^{\infty} E_i\Big) \quad m\Big(\bigcup_{i=1}^{n} E_i\Big), \text{ for any n.}$$

Therefore,

$$m\Big(\bigcup_{i=1}^{\infty} E_i\Big) \quad \sum_{i=1}^{n} mE_i, \text{ for any n.} \qquad \text{[using (2)]}$$

Therefore,

$$m\Big(\bigcup_{i=1}^{\infty} E_i\Big) \quad \sum_{i=1}^{\infty} m(E_i). \qquad \ldots(3)$$

Since m* is countably subadditive ,

m is also countably subadditive on M .

Therefore,

$$m\Big(\bigcup_{i=1}^{\infty} E_i\Big) \quad \sum_{i=1}^{\infty} m(E_i). \qquad ..(4)$$

From (3) and (4),

$$m\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \sum_{i=1}^{\infty} m(E_i).$$

Hence the theorem.

Theorem 16.14

Let $\{E_n\}$ be an infinite decreasing sequence of measurable sets.

i.e $E_{n+1} \subset E_n$ for each n.

Let mE_1 be finite.

Then $m\Big(\bigcap_{i=1}^{\infty} E_i\Big) = \lim_{n›}\ mE_n$.

Proof

Let $E = \bigcap_{i=1}^{\infty} E_i$.

Let $F_i = E_i - E_{i+1}$.

Claim: (i) $E_1 - E = \bigcup_{i=1}^{\infty} F_i$ and

(ii) F_i's are pairwise disjoint.

Let $x \in E_1 - E$.

Then $x \in E_1$

and $x \notin E = \bigcap_{i=1}^{\infty} E_i$.

i.e $x \in E_1$ and $x \notin E_i$ for some i.

Since $\{E_n\}$ is an infinite decreasing sequence of measurable sets,

$E_1 \supset E_2 \supset \ldots . E_i$, for all i.

Let j be the smallest suffix for which $x \notin E_j$.
i.e $x \in E_{j-1}$ but $x \notin E_j$.
Therefore,

$$x \in E_{j-1} - E_j = F_{j-1}.$$

Therefore,

$$x \in \bigcup_{i=1}^{\infty} F_i .$$

Hence,

$$E_1 - E \subset \bigcup_{i=1}^{\infty} F_i . \qquad \ldots(1)$$

Let $y \in \bigcup_{i=1}^{\infty} F_i$.
This implies,

$y \in F_i$ for some i.

This implies,

$y \in E_i$ but $y \notin E_{i+1}$.

Since $E_1 \supset E_i$,

$y \in E_1$.

Since $y \notin E_{i+1}$.

$$y \notin \bigcap_{i=1}^{\infty} E_i = E.$$

Therefore ,

$y \in E_1 - E$.

Hence,

$$\bigcup_{i=1}^{\infty} F_i \subset E_1 - E. \qquad \ldots(2)$$

From (1) and (2),

$$E_1 - E = \bigcup_{i=1}^{\infty} F_i .$$

Hence (i) is proved.

To prove (ii) of the claim
i.e F_i's are pairwise disjoint, where $F_i = E_i - E_{i+1}$.

Let i j.
Without loss of generality, assume that $i < j$.
Then $E_i \supset E_j$.
Let $x \in F_i$.
This implies,

$x \in E_i$ and $x \notin E_{i+1}$.

Since $E_{i+1} \supset E_i \supset E_j$,
this implies,

$x \notin E_j$

Therefore,

$x \notin F_j$.

Therefore,

$F_i \not\subset F_j$.

Similarly we can prove that $F_j \not\subset F_i$.
Hence the F_i's are pairwise disjoint.
Hence our claim is proved.

Now let us prove the theorem.
Since E_i, and E_{i+1} are measurable,
$F_i = E_i - E_{i+1}$ is measurable, for all i.

Also $E_1 - E = \bigcup_{i=1}^{\infty} F_i$ and

$F_i \cap F_j = \varphi$, for $i \quad j$.

Therefore ,

$$m(E_1 - E) = m(\bigcup_{i=1}^{\infty} F_i)$$

$$= \sum_{i=1}^{\infty} m(F_i)$$

$$= \sum_{i=1}^{\infty} m(E_i - E_{i+1}) \qquad \ldots(3)$$

Since $E_1 \supset E$,
$E_1 = E \cup (E_1 - E)$ is a disjoint union.
Since E_1 and E are measurable,
$E_1 - E$ is measurable.
Therefore,

$$m(E_1) = m(E) + m(E_1 - E) \qquad \ldots(4)$$

Similarly ,
since $E_i \supset E_{i+1}$,

$$m(E_i) = m(E_{i+1}) + m(E_i - E_{i+1}) \qquad \ldots(5)$$

Since $E_i \subset E_1$, for all i,
$m(E_i) \quad m(E_1)$, for all i.
Given $m(E_1) <$.
Therefore ,
$m(E_i) <$, for all i.

From(4),

$$m(E_1 - E) = m(E_1) - m(E) \qquad \ldots(6)$$

from (5),

$$m(E_i - E_{i+1}) = m(E_i) - m(E_{i+1}) \qquad \ldots(7)$$

Therefore,

$$m(E_1) - m(E) = m(E_1 - E)$$

$$= \sum_{i=1}^{\infty} m(E_i - E_{i+1}) \qquad [\text{from}(3)]$$

$$= \sum_{i=1}^{\infty} (m(E_i) - m(E_{i+1})) \qquad [\text{from } (7)]$$

$$= \lim_{n›} \sum_{i=1}^{n-1} (m(E_i) - m(E_{i+1}))$$

$$= \lim_{n›} (mE_1 - mE_n)$$

$$= mE_1 - \lim_{n›} mE_n.$$

Hence,

$$m(E_1) - m(E) = mE_1 - \lim_{n›} \quad mE_n.$$

Since $m(E_1) <$, canceling it both sides, we get

$$m(E) = \lim_{n›} \quad mE_n.$$

$$\text{i.e } m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n›} \quad mE_n.$$

Hence the theorm.

Theorem16.15 Littlewood's First Principle

Let E be a given set.

Then the following statements are equivalent:

(i) E is measurable

(ii) Given $\varepsilon > 0$, there is an open set $U \supset E$ with $m^*(U - E) < \varepsilon$.

(iii) Given $\varepsilon > 0$, there is a closed set $F \subset E$ with $m^*(E - F) < \varepsilon$.

(iv) There is a G_δ set G with $E \subset G$ such that $m^*(G - E) = 0$.

(v) There is a F_σ set F with $F \subset E$ such that $m^*(E - F) = 0$

If m^*E is finite, the above statements are equivalent to

(vi) Given $\varepsilon > 0$, there is a finite union V of open intervals such that

$$m^*(V \Delta E) < \varepsilon .$$

Proof

To prove that (i) implies (ii).

Let E be measurable.

Case.1:Suppose $m^*E = mE <$.

Then given $\varepsilon > 0$, there exist open intervals I_n, n=1,2,3,…

Such that $E \subset \bigcup_{n=1}^{\infty} I_n$ and

$$l(I_n) \quad m^*E + \varepsilon$$

(or) $m^*E + \varepsilon \quad l(I_n)$ …(1)

Let $U = \bigcup_{n=1}^{\infty} I_n$.

Then U is an open set, [since each I_n is open.]

$U \supset E$ and

U is measurable. [since each I_n is measurable]

By the countable subadditivity of m,

$$m^*U = m\left(\bigcup_{n=1}^{\infty} I_n\right)$$

$$= \sum_{n=1}^{\infty} m(I_n)$$

$$= \sum_{n=1}^{\infty} l(I_n) \qquad \text{…(2)}$$

From (1) and (2),

$$m^*E + \varepsilon \quad m^*U \qquad \text{…(3)}$$

Also $U = E \cup (U - E)$ is a disjoint union.

Therefore,

$$m^*U = m^*E + m^*(U - E)$$

Therefore ,

$$m^*(U - E) = m^*U - m^*E \quad \varepsilon \qquad \text{[by (3)]}$$

Hence there is an open set $U \supset E$ with $m^*(U - E) < \varepsilon$.

Case .2 Suppose $m^*E = mE = \ .$

Let $X = \bigcup_{n=0}^{\infty} I_n$ where $I_n = [n, n+1)$

$Y = \bigcup_{n=0}^{\infty} I_n'$ where $I_n' = (-n-1, -n]$.

Then $R = X \cup Y$.

Now, I_n , I_n' are countable union of disjoint finite intervals.

Rename their union as $\{J_n\}_{n=1}^{\infty}$.

Hence

$$\begin{aligned} E &= E \cap R \\ &= E \cap \bigcup_{n=1}^{\infty} J_n \\ &= \bigcup_{n=1}^{\infty} (E \cap J_n) \\ &= \bigcup_{n=1}^{\infty} E_n, \quad \text{where } E_n = E \cap J_n . \end{aligned}$$

Since E and J_n are measurable, E_n's are also measurable.

Also $E_n \subset J_n$.

Hence $mE_n \quad mJ_n < \ .$

Hence by case.1 ,

we can find open sets U_n such that $E_n \subset U_n$

and $m(U_n - E_n) < \varepsilon/2^n$.

Let $U = \bigcup_{n=1}^{\infty} U_n$.

Then

$$\begin{aligned} U - E &= \bigcup_{n=1}^{\infty} U_n - \bigcup_{n=1}^{\infty} E_n \\ &\subset \bigcup_{n=1}^{\infty} (U_n - E_n) \end{aligned}$$

Therefore,

$$\begin{aligned} m(U - E) \quad & m\Big(\bigcup_{n=1}^{\infty} (U_n - E_n)\Big) \\ & \sum_{n=1}^{\infty} m(U_n - E_n) \\ & < \sum_{n=1}^{\infty} \varepsilon / 2^n \\ & = \varepsilon. \end{aligned}$$

Hence,

$$m^*(U - E) < \varepsilon.$$

From case .1 and 2,
we get that (i) implies (ii).

To Prove that (ii) implies (iv):

Assume that for every $\varepsilon > 0$,
there exists an open set $U \supset E$ with $m^*(U - E) < \varepsilon$.
Hence for $\varepsilon = 1/n$, $n=1,2,3,\ldots,$
there exist open sets $U_n \supset E$ with $m^*(U_n - E) < 1/n$.

Let $G = \bigcap_{n=1}^{\infty} U_n$.

Then $G \supset E$ and
Since G is a countable union of open sets ,
G is a G_δ set.
Since $G \subset U_n$, for every n,
$G - E \subset U_n - E$, for every n.
Therefore,
$m^*(G - E) \quad m^*(U_n - E) < 1/n$, for every n.
Therefore,
$m^*(G - E) = 0$.
Hence,
there is a G_δ set G with $E \subset G$ such that $m^*(G - E) = 0$.
Hence (ii) implies (iv).

To prove that (iv) implies (i):
Assume that there is a G_δ set G with $E \subset G$ such that $m^*(G - E) = 0$.
Since any G_δ set is measurable ,
G is measurable.
Since $m^*(G - E) = 0$,
$G - E$ is measurable.
Hence $E = G - (G - E)$ is measurable.
Hence (iv) implies (i).

Hence it follows that ,
(i) => (ii) =>(iv) =>(i).
Hence we get that,
(i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv). …(1)

To prove that (ii) implies (iii):
Assume (ii)
i.e Given $\varepsilon > 0$, there is an open set $U \supset E$ with $m^*(U - E) < \varepsilon$.
From(1),
(ii) implies (i).
Hence E is measurable.
Hence $\tilde{E}$ is measurable.
Hence for given given $\varepsilon > 0$,
there is an open set $U \supset \tilde{E}$ with $m^*(U - \tilde{E}) < \varepsilon$. [Since (i) implies (ii).]
Since $U \supset \tilde{E}$,

$\tilde{U} \subset E$ and
$U - \tilde{E} = E - \tilde{U}$
$= U \cap E.$
Since U is open,
$\tilde{U}$ is closed.
Also $m^*(E - \tilde{U}) = m^*(U - \tilde{E}) < \varepsilon$.
Hence,
there is a closed set $\tilde{U} \subset E$ with $m^*(E - \tilde{U}) < \varepsilon$.
Hence (ii) implies (iii)

To prove that (iii) implies (v):
Assume (iii)
i.e Given $\varepsilon > 0$, there is a closed set $F \subset E$ with $m^*(E - F) < \varepsilon$.

Take $\varepsilon = 1/n$.
Then for each n,
there is a closed set $F_n \subset E$ with $m^*(E - F_n) < 1/n$.
Let $F = \bigcup_{n=1}^{\infty} F_n$.
Since F is a countable union of closed sets ,
F is a F_σ set.
Also $F_n \subset E$, for every n.
Hence ,
$F = \bigcup_{n=1}^{\infty} F_n \subset E$ and
$E - F \subset E - F_n$, for every n.
Hence,
$m^*(E - F) \quad m^*(E - F_n) < 1/n$, for every n.
Hence,
$m^*(E - F) = 0$.
Hence,
there is a F_σ set F with $F \subset E$ such that $m^*(E - F) = 0$.
Hence (iii) implies (v)

To prove that (v) implies (i).
Assume (v)
i.e There is a F_σ set F with $F \subset E$ such that $m^*(E - F) = 0$.

Since each F_σ set is measurable,
F is measurable.
Since $m^*(E - F) = 0$,
$E - F$ is measurable.
Hence ,
$E = F \cup (E - F)$ is measurable.
Hence (V) implies (i).
Hence we have shown that,
(ii) =>(iii) =>(v) =>(i) …(2)
From (1) and (2),

it is clear that (i) to (v) are equivalent.

Hence the theorem.

Check your progress

1.Prove that properties (i) to (v) are equivalent to (vi), if m*E is finite.

2. Show that if E is measurable, then each translate E+y is also measurable.

3.Show that if E_1 and E_2 are measurable,

then $m(E_1 \bigcup E_2)+m(E_1 \bigcap E_2)=mE_1+mE_2$.

4.Let $\{E_i\}$ be a sequence of disjoint measurable sets and A be any set.

Show that $m^*(A \cap \bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m^*(A \bigcap E_i)$

16.5 Let us sum up

Thus in this lesson , we have seen

- The definition of outer measure of sets
- Outer measure of an interval is its length
- Some important properties of Outer measure
- The definition of Measurable sets
- Countable union of measurable sets is also measurable
- Countable intersection of measurable sets is also measurable
- Every Borel set is measurable and
- Littlewood's First Principle.

16.6 Lesson End Activities

1. Prove that m^* is translation invariant

2. If E_1 and E_2 are measurable sets, Prove that $m(E,VE_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$

3. Prove that m^* (A) = 0 if A is a countable set.

16.7 References

1. Real Analysis by H.L. Royden (third Edition)

2. Measure theory and Intergration by G. deBarra.

LESSON – 17

MEASURABLE FUNCTIONS AND LITTLEWOOD'S THEOREM

Contents

17.0 Introduction

Already we have studied Littlewood's first principle. In this lesson we are going to study about measurable functions and littlewood's second and third principles.

17.1 Aims and objectives

After studying this lesson, you would know

- What are measurable functions?
- Sum, difference ,scalar product and product of measurable functions are measurable and
- Littlewood's Theorems

17.2 Measurable functions

Theorem17.1

Let f be an extended real valued function whose domain is measurable.
Then the following statements are equivalent:

(i) For each real number α, the set $\{x \ /f(x) > \alpha\}$ is measurable.
(ii) For each real number α, the set $\{x \ /f(x) \geq \alpha\}$ is measurable.
(iii) For each real number α, the set $\{x \ /f(x) < \alpha\}$ is measurable.
(iv) For each real number α, the set $\{x \ /f(x) \leq \alpha\}$ is measurable.

These statements imply

(v) For each real number α, the set $\{x \ /f(x) = \alpha\}$ is measurable.

Proof

Let the domain of f be D.
Then D is measurable.

To prove that (i) implies (iv)

Let $\{x \ /f(x) > \alpha\}$ is measurable, for every real α.
Since $\{x \ /f(x) \leq \alpha\} = D - \{x \ /f(x) > \alpha\}$ and
difference of two measurable sets is measurable,
$\{x \ /f(x) \leq \alpha\}$ is measurable.
Hence (i) implies (iv)

To prove that (iv) implies (i)

Assume(iv).i.e $\{x \ /f(x) \leq \alpha\}$ is measurable, for every real α.
Then $\{x \ /f(x) > \alpha\} = D - \{x \ /f(x) \leq \alpha\}$

being the difference of two measurable sets is measurable.
Hence (iv) implies (i)

Hence (i) ⇔ (iv) (1)

To prove that (ii) implies (iii)

Assume(ii) i.e {x /f(x)   α } is measurable, for every real α.
Then {x /f(x) < α }=D -{x /f(x)   α }
being the difference of two measurable sets is measurable.
Hence (ii) implies (iv)

To prove that (iii) implies (ii)
Assume (iii)i.e {x /f(x) < α } is measurable, for every real α.
Then {x /f(x)   α } = D -{x /f(x) < α }
being the difference of two measurable sets is measurable.
Hence (iii) implies (ii)

Hence (ii) ⇔ (iii) …(2)

To prove that (i) implies (ii)

Assume (i) i.e{x /f(x) > α } is measurable, for every real α.
Therefore {x / f(x) > α -1/n} is measurable, for n=1,2,3,..
Also countable intersection of measurable sets is measurable.
Hence ,

{x /f(x)   α } = $\bigcap_{n=1}^{\infty}\{x/\,f(x)>\alpha-\frac{1}{n}\}$ is measurable.

Hence (i) implies (ii)

To prove that (ii) implies (i)

Assume (ii)i.e{x /f(x)   α } is measurable, for every real α.
Then for n=1, 2, 3,…,
{x /f(x)   α+1/n} is measurable.
Also countable union of measurable sets is measurable.
Hence,

{x /f(x) > α }=$\bigcup_{n=1}^{\infty}\{f(x)\geq\alpha+\frac{1}{n}\}$ is measurable.

Hence (ii) implies (i).

Hence (i) ⇔ (ii) …(3)

From(1), (2) and (3),
(iv) ⇔ (i) ⇔ (ii) ⇔ (iii)
Hence the first four statements are equivalent.

To prove (v):

Case1: α .
Then {x/ f(x) =α}={x /f(x) α } $\bigcap$ {x /f(x) α }.
Since intersection of two measurable sets is measurable ,
{x/ f(x) =α} is measurable.

Case2: α = .
Then {x/ f(x) =α} = {x/ f(x) = }

$$=\bigcap_{n=1}^{\infty}\{x/\, f(x) \geq n\}.$$

By assumptions {x /f(x) n} is measurable for every n=1,2,3….
Also countable intersection of measurable sets is measurable.
Hence {x/ f(x) = } is measurable.

Case3. α = - .

Then {x/ f(x) = - } = $\bigcap_{n=1}^{\infty}\{x/\, f(x) \leq -n\}$

By assumptions {x/f(x) -n} is measurable, for n=1,2,3…
Also countable intersection of measurable sets is measurable.
Hence {x/ f(x) = - } is measurable.

Hence the set {x /f(x) = α } is measurable,for each real α.
Hence (v) follows from(i),(ii),(iii) and (iv).
Hence the theorem.

Definition

An extended real valued function f is Lebesgue measurable if its domain is measurable and if it satisfies one of the following four statements:

(i) For each real number α, the set {x /f(x) > α } is measurable.
(ii) For each real number α, the set {x /f(x) α } is measurable.
(iii) For each real number α, the set {x /f(x) < α } is measurable.
(iv) For each real number α, the set {x /f(x) α } is measurable.

Check your progress

1. Show that constant functions are measurable.
2. Show that if A is measurable subset of R, then its characteristic function χ_A is a measurable function.
3. Show that continuous functions from R› R are measurable.
4. Show that any step function is measurable.

Answers

1. Let f : D › R $\bigcup$ { - , } be defined as
f(x) =c, for every x ∈ D,
where D is a measurable subset of R.

Now If α < c,
then {x /f(x) > α }=D
And If α c,
then {x /f(x) > α }=φ.

Since both D and φ are measurable functions,
f is also a measurable function.
Hence constant functions are measurable.

2. $$\chi_A(x) = \begin{cases} 1 & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases}$$

Therefore,

$$\{x \ / \chi_A(x) > \alpha \} = \begin{cases} R & if \quad \alpha < 0 \\ A & if \quad 0 \le \alpha < 1 \\ \varphi & if \quad \alpha \ge 1 \end{cases}$$

Since R, A, and φ are measurable functions,
χ_A is also a measurable function.

3. Assume that f is a continuous function from R to R.
Then $\{x \ /f(x) > \alpha \} = f^{-1}((\alpha, \ \ ))$
Since $(\alpha, \ )$ is open in R and f is continuous ,
$f^{-1}((\alpha, \ \ ))$ is open in R.
Since any open set in R is measurable ,
$f^{-1}((\alpha, \ \ ))$ is also measurable.
Hence $\{x \ /f(x) > \alpha \}$ is measurable.
Hence f is measurable.
Hence continuous functions from R› R are measurable.

4.A real valued function φ defined on [a, b] is called a step function if there is a partition
$a = x_0 < x_2 < \ldots . x_n = b$,
such that for each i, the function assumes only one value in the interval (x_i , x_{i+1}).
Suppose if $\varphi(x_{i-1} , x_i) = \alpha_{i-1}$, i=1,2,…n.

$$\text{Then} \quad \{ x \ / \varphi(x) > \alpha \} = \begin{cases} \phi & if \quad \alpha \ge \alpha_{i-1}, i = 1,2,\ldots n \\ [a,b] & if \quad \alpha < \alpha_{i-1}, i = 1,2,\ldots n \\ a & finite \ union \ of \ open \ intervals \quad if \quad \alpha \end{cases}$$
lies between min and max of $\{\alpha_0 , \alpha_1, \ \ .\alpha_n\}$.

since ϕ , [a, b] and finite union of open intervals are all measurable,
φ is also measurable.

Theorem 17.2

Let c be a constant and f and g are two measurable real valued functions defined on the same domain. Then the functions f+c, cf, f+g, f-g and fg are also measurable.

Proof:

(i)To show that f+c is measurable.

Consider $\{ x / f(x)+c > \alpha \} = \{ x / f(x) > \alpha - c \}$.
Since f is measurable,
$\{ x / f(x) > \alpha - c \}$ is measurable.
Hence ,
$\{ x / f(x)+c > \alpha \}$ is measurable.

Hence f+c is measurable.

(ii)To show that cf is measurable:

Case1: c>0

Consider $\{ x / (cf)(x) > \alpha \} = \{ x / cf(x) > \alpha \}$

$= \{ x / f(x) > \alpha/c \}$

Since f is mesurable,

$\{ x / f(x) > \alpha/c \}$ is measurable.

Hence $\{ x / (cf)(x) > \alpha \}$ is measurable.

Hence cf is measurable, for all real constants c.

Case2: c=0.

Then cf= 0 is a constant function.

Hence cf is measurable.

Case3: c<0.

Then $\{ x / (cf)(x) > \alpha \} = \{ x / f(x) < \alpha/c \}$.

Since f is mesurable,

$\{ x / f(x) < \alpha/c \}$ is measurable.

Hence $\{ x / (cf)(x) > \alpha \}$ is measurable.

Hence cf is measurable.

(iii)To show that f+g is measurable.

Consider $\{x /(f+ g)(x) < \alpha\} = \{x /f(x) +g(x) < \alpha\}$

$= \{x /f(x) < \alpha - g(x)\}$.

Since between any two reals, there is a rational,

there exists a rational r such that $f(x) < r < \alpha - g(x)$.

Therefore ,

$$\{x /(f+ g)(x) < \alpha\} = \bigcup_r [\{x / f(x) < r\} \cap (x / g(x) < \alpha - r \}].$$

Since f and g are measurable,

$\{ x / f(x) < r \}$ and $\{x / g(x) < \alpha - r\}$ are measurable.

Hence,

$\{ x / f(x) < r \} \cap \{x / g(x) < \alpha - r\}$ is measurable.

Since the rationals are countable and

countable union of measurable sets is measurable,

$\{x /(f+ g)(x) < \alpha\}$ is measurable.

Hence f+g is measurable.

(iv)To show that f – g is measurable.

Now $f - g = f+(-1)g$.

From(ii),

(-1)g is measurable.

Hence from (iii),

$f + (-1)g$ is measurable.

Hence $f - g$ is measurable.

(v)To show that f^2 is measurable.

Consider $\{x / f^2(x) > \alpha\}$

$$= \{x / f(x) > \sqrt{\alpha}\} \cup \{x / f(x) < -\sqrt{\alpha}\} \quad \text{when } \alpha \quad 0.$$
$$= D, \qquad \text{when } \alpha < 0.$$

Since f is measurable,

$\{x / f(x) > \sqrt{\alpha}\}$ and $\{x / f(x) < -\sqrt{\alpha}\}$ are measurable.

Hence,

$\{x / f(x) > \sqrt{\alpha}\} \cup \{x / f(x) < -\sqrt{\alpha}\}$ is measurable.

Since D is the domain of f,

D is also measurable.

Hence $\{x / f^2(x) > \alpha\}$ is measurable.

Hence f^2 is measurable.

(vi)To show that fg is measurable.

From (iii) and (iv),

$f+g$ and $f-g$ are measurable.

Hence by (v),

$(f+g)^2$ and $(f-g)^2$ are measurable.

Hence again by(iv),

$[(f+g)^2 - (f-g)^2]$ is measurable.

Hence by(ii),

$(1/4)[(f+g)^2 - (f-g)^2]$ is measurable.

Hence fg is measurable.

Hence the theorem.

Theorem17.3

Let $f_1, f_2, \ldots. f_n$ be measurable functions defined on the same domain D. Then $\max(f_1, f_2, \ldots. f_n)$ and $\min(f_1, f_2, \ldots. f_n)$ are measurable.

Proof

Let $h = \max(f_1, f_2, \ldots. f_n)$

Therefore ,

$h(x) = \max(f_1(x), f_2(x), \ldots. f_n(x))$, for all $x \in D$.

Therefore ,

$$\{x / h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x / f_i(x) > \alpha\} \ .$$

Since each f_i is measurable,

$\{x / f_i(x) > \alpha\}$ is measurable, for $i=1,2,\ldots,n$, and for every real α.

Since finite union of measurable sets is also measurable,

$\{x / h(x) > \alpha\}$ is measurable.

Hence $h = \max(f_1, f_2, \ldots. f_n)$ is measurable.

Let $g = \min(f_1, f_2, \ldots f_n)$

Therefore,

$g(x) = \min(f_1(x), f_2(x), \ldots f_n(x))$, for all $x \in D$.

Therefore,

$$\{x / g(x) < \alpha\} = \bigcup_{i=1}^{n} \{x / f_i(x) < \alpha\}$$

Since each f_i is measurable,

$\{x / f_i(x) < \alpha\}$ is measurable, for i=1,2,…n, and for every real α.

Since finite intersection of measurable sets is also measurable,

$\{x / g(x) < \alpha\}$ is measurable.

Hence $g = \min(f_1, f_2, \ldots f_n)$ is measurable.

Hence the theorem.

Remarks:

From the above theorem, we get the following important results.

1. If f and g are measurable, then $f \vee g = \max(f, g)$ is measurable.
2. If f and g are measurable, then $f \wedge g = \min(f, g)$ is measurable.
3. If f is measurable, $f^+ = f \vee 0 = \max(f, 0)$ is measurable.
4. f^+ is called the positive part of f.
5. If f is measurable, $f^- = f \wedge 0 = \min(f, 0)$ is measurable.
6. f^- is called the negative part of f.
7. If f^+ and f^- are measurable, then $f = f^+ - f^-$ is measurable.
   and $|f| = f^+ + f^-$ is measurable.
8. From 3,5and 7, it is clear that if f is measurable, then |f| is measurable.
9. But the converse is not true. i.e if |f| is measurable, then f need not be measurable.

Theorem17.4

Let $\{f_n\}$ be a sequence of measurable functions with the same domain of definition. Then the functions $\sup_n f_n$, $\inf_n f_n$, $\overline{\lim} f_n$, $\underline{\lim} f_n$ are all measurable.

Proof

Let $\{f_n\}$ be a sequence of measurable functions with the same domain of definition.

To prove that $\sup_n f_n$ is measurable.

Let $g(x) = \sup_n f_n(x)$.

Consider $\{x / g(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x / f_n(x) > \alpha\}$.

Since each f_n is measurable,

$\{x / f_n(x) > \alpha\}$ is measurable, for each n, and for every real α.

Also countable union of measurable sets is measurable.

Hence $\{x / g(x) > \alpha\}$ is measurable.

Hence $g = \sup_n f_n$ is measurable. …(1)

To prove that $\inf_n f_n$ is measurable.

Let $h(x) = \inf_{n} f_n(x)$.

Consider $\{x / h(x) < \alpha\} = \bigcup_{n=1}^{\infty} \{x / f_n(x) < \alpha\}$.

Since each f_n is measurable,

$\{x / f_n(x) < \alpha\}$ is measurable, for each n, and for every real α.

Also countable intersection of measurable sets is measurable.

Hence $\{ x / h(x) < \alpha \}$ is measurable.

Hence $h = \inf_{n} f_n$ is measurable. …(2)

To prove that $\overline{\lim} f_n$ is measurable.

Now $\overline{\lim} f_n = \inf_{n} (\sup_{k \geq n} f_k) = \inf_{n} g_n$.

Since each f_k is measurable,

$g_n = \sup_{k \geq n} f_k$ is measurable, for every n. [from(1)]

Therefore $\inf_{n} g_n$ is measurable. [from (2)]

Hence $\overline{\lim} f_n$ is measurable.

To prove that $\underline{\lim} f_n$ is measurable.

Similarly $\underline{\lim} f_n = \sup_{n}(\inf_{k \geq n} f_k) = \sup_{n} h_n$

Since each f_k is measurable,

$h_n = \inf_{k \geq n} f_k$ is measurable, for every n. [from(2)]

Hence $\sup_{n} h_n$ is measurable. [from (2)]

Hence $\underline{\lim} f_n$ is measurable.

Hence the theorem.

Remarks

.Let $\{ f_n \}$ be a converging sequence of measurable functions on the same domain D. Let $\lim_{n›} f_n = f$. Then f is measurable.

Proof

Since $\lim_{n›} f_n$ exists ,

$$\overline{\lim} f_n = \underline{\lim} f_n = \lim_{n›} f_n.$$

By the above theorem ,

both $\overline{\lim} f_n$ and $\underline{\lim} f_n$ are measurable.

Hence $\lim_{n›} f_n$ is measurable.

Hence f is measurable.

Definition

A property is said to hold at almost everywhere if the set of points at which the property fails to hold is a set of measure is zero.

Thus, in particular,

f = g a.e if f and g have the same domain D and

$$m\{ x / f(x) \quad g(x) \} = 0.$$

Definition

If $\{ f_n \}$ is a sequence of functions, then $\{ f_n \}$ is said to converge to g almost everywhere , if there is a set E of measure zero such that $\{f_n(x)\}$converges to g(x) for all x not in E.

Theorem17.5

If f is measurable and f= g a.e , then g is measurable.

Proof

Let $E = \{ x / f(x) \quad g(x) \}$

Since f = g a.e, mE=0.

Now ,

$$\{ x / g(x) > \alpha \}$$

$$= \{x / f(x) > \alpha \} \cup \{x \in E / g(x) >\alpha\} - \{ x \in E/ g(x) \quad \alpha \}.$$

Since $\{x \in E / g(x) >\alpha\} \subset E$

and $\{ x \in E/ g(x) \quad \alpha \} \subset E$

and mE= 0, we get that

measures of $\{x \in E / g(x) >\alpha\}$ and $\{ x \in E/ g(x) \quad \alpha \}$ are zero.

Hence,

$\{x \in E / g(x) >\alpha\}$ and $\{ x \in E/ g(x) \quad \alpha \}$ are measurable.

Since f is measurable,

$\{x / f(x) > \alpha \}$ is measurable.

Hence $\{x / g(x) > \alpha \}$ is measurable, for every real α.

Hence g is measurable.

Hence the theorem.

Theorem 17.6 Littlewood's second principle

Let f be a measurable function defined on an interval [a, b], and assume that f takes values ± only on a set of measure zero.

Then given ε >0, we can find a step function g and a continuous function h such that

$$| f - g | < \varepsilon \quad \text{and} \quad | f - h | < \varepsilon.$$

Check your progress

Prove theorem 17.6 using the following results.

(a) Given a measurable function f on [a, b] that takes values ± only on a set of measure zero, and given ε >0,there is an M such that $|f| \quad M$ except on a set of measure less than ε/3.

(b) Let f be a measurable function on [a, b].Given ε >0 and M, there is a simple function φ such that $|f(x) - \varphi(x)| < \varepsilon$ except where $|f(x)| \quad M$. If $m \quad f \quad M$, then wee may take φ so that $m \quad \varphi \quad M$.

(c) Given a simple function φ on [a, b], there is a step function g on [a, b] such that $g(x) = \varphi(x)$ except on a set of measure less than ε/3. If $m \quad \varphi \quad M$, we can take g so that $m \quad g \quad M$.

(d) Given a step function g on [a, b] , there is a continuous function h such that $g(x)=h(x)$ except on a set of measure less than ε/3. If $m \quad g \quad M$, we can take h so that $m \quad h \quad M$.

17.3 Littlewood's theorem

Theorem17.7 Littlewood's third principle

Let E be a measurable set of finite measure, and { f_n } a sequence of measurable functions defined on E. Let f be a real valued function such that for each x in E we have $f_n(x)$› f (x) .

Then given ε >0 and δ > 0, there is a measurable set A ⊂ E with mA < δ and an integer N such that for all x ∉ A and all n N,

$|f_n(x) – f(x)| < ε$.

Proof

Let G_n ={x∈E / $|f_n(x) – f(x)|$ ε}

Let $E_N = \bigcup_{n=N}^{\infty} G_n$

$= \{x \in E / |f_n(x) – f(x)|$ ε, for some n N}

Since $E_{N+1} = \bigcup_{n=N+1}^{\infty} G_n$

$\subset \bigcup_{n=N}^{\infty} G_n$

= E_n,

{ E_N }is a decreasing sequence of sets.

Since $f_n(x)$›f(x), for every x ∈E,

for each x ∈E, there exists some E_N such that

x ∉ E_N.

This implies that,

$\bigcap E_N = \phi$.

Since f and f_n are measurable functions,

f $_n$–f is measurable, for every n.

This implies that ,

| f $_n$–f | is measurable, for every n.

Therefore,

G_n is measurable , for every n.

Since E_N is a union of these measurable sets,

E_N is measurable, for every n.

Since E is of finite measure and $E_N \subset E$,

E_N's are also of finite measure.

Therefore by the theorem 16.14,

"Let {E_n } be an infinite decreasing sequence of measurable sets. i.e $E_{n+1} \subset E_n$ for each n. Let mE_1 be finite.

Then $m(\bigcap_{i=1}^{\infty} E_i) = \lim_{n›} mE_n$." ,

$\lim_{n›} mE_n = m(\bigcap_{i=1}^{\infty} E_i)$

$= m\phi$

= 0.

Hence given δ > 0,there exists N such that

$mE_N < δ$ and

m({x∈E / $|f_n(x) – f(x)|$ ε, for some n N}) < δ.

Let this E_N be denoted by A.

Then mA < δ

And $\tilde{A} = \{x \in E \,/\, |f_n(x) - f(x)| < \varepsilon$, for all $n \quad N\}$.
Hence there is a measurable set $A \subset E$ with $mA < \delta$ and an integer N such that for all $x \notin A$ and all $n \quad N$,

$$|f_n(x) - f(x)| < \varepsilon .$$

Hence the theorem.

17.4 Let us sum up

Thus in this lesson , we have seen

- The definition of measurable function
- Constant functions, Characteristics functions, continuous functions, step functions on a measurable domain are measurable functions.
- Sum, difference , product and scalar multiple of measurable functions are measurable functions.
- max $(f_1, f_2, \ldots f_n)$ and $\min(f_1, f_2, \ldots f_n)$ are measurable if $f_1, f_2, \ldots f_n$ are measurable .
- $\sup_n f_n$, $\inf_n f_n$, $\overline{\lim} f_n$, $\underline{\lim} f_n$ of a sequence of measurable functions are measurable.
- The definition of almost everywhere and almost everywhere convergence and some theorems based on them and
- Littlewood's principles

17.5 Lesson End Activities

1. D is a dense subset of the set of all real numbers. Set f be an extended real – valued function on R such that $\{ x : f(x) > \propto \}$ is measurable for each $\propto \in D$. prove that f is a measurable function.

2. if $\{fn\}^{\propto}_{n-1}$ is a sequence of measurable functions, prove that the set of points x at which $\{fn(x)\}^{\propto}_{n-1}$ converges is a measurable set.

17.6 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

LESSON 18

THE LEBESGUE INTEGRAL OF BOUNDED FUNCTIONS OVER A SET OF FINITE MEASURE

Contents

18.0 Introduction

In this lesson, we are going to study the Lebesgue integrals of a simple function and bounded functions.

18.1 Aims and objectives

After studying this lesson, you would know

- The definition of a simple function
- The Lebesgue integral of a simple function which vanish outside a set of finite measure.
- Some of its properties
- The Lebesgue integral of bounded functions over a set of finite measure.
- Properties of the Lebesgue integral of bounded functions over a set of finite measure and
- Bounded convergence theorem.

18.2 Lebesgue integral of a simple function which vanish outside a set of finite measure

Definition

The function χ_E defined by

$$\chi_E(x) = \begin{cases} 1 & if \quad x \in E \\ 0 & if \quad x \notin E \end{cases}$$

is called the characteristic function of E.

Definition

A linear combination

$$\varphi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

is called a simple function if the sets E_i are measurable.

Remark

1. This representation of φ is not unique.
2. A function φ is simple if and only if it is measurable and assumes only a finite number of values.
3. If φ is a simple function and $\{a_1, a_2, \ldots a_n\}$ the set of non-zero values of φ, then

$$\varphi = \sum a_i \chi_{A_i}$$

where A_i ={ x / φ(x) = a_i}.

This representation of φ is called the canonical representation, where the A_i's are disjoint and the a_i's are distinct and non-zero.

Definition

If φ vanishes outside a set of finite measure and φ has the canonical representation

$$\varphi = \sum_{i=1}^{n} a_i \chi_{A_i} ,$$

then we define the integral of φ by

$$\int \varphi(x)dx = \sum_{i=1}^{n} a_i mA_i .$$

Remark:

4. The expression for the integral of φ is sometimes abbreviated as $\int \varphi$.
5. If E is any measurable set, we define

$$\int_E \varphi = \int \varphi \cdot \chi_E$$

Theorem 18.1

Let $\varphi = \sum a_i \chi_{E_i}$, with $E_i \bigcap E_j = \phi$ for i j .

Suppose each set E_i is a measurable set of finite measure. Then

$$\int \varphi = \sum_{i=1}^{n} a_i mE_i .$$

Proof

The set A_a ={ x / φ(x) = a} = $\bigcup_{a_i = a} E_i$.

Hence $mA_a = \sum_{a_i = a} a_i mE_i$ (by the additivity of m)

Therefore ,

$$\int \varphi(x)dx = \sum amA_a$$
$$= \sum_{i=1}^{n} a_i mE_i .$$

Hence the theorem.

Theorem 18.2

Let φ and ψ be simple functions which vanish outside a set of finite measure . Then

$$\int (a\varphi + b\chi) = a\int \varphi + b\int \chi ,$$

and if φ ψ a.e., then

$$\int \varphi \qquad \int \chi .$$

Proof

Let {A_i } and {B_i } be the sets occurring in the canonical representation of φ and ψ .

Let A_0 and B_0 be the sets where φ and ψ are zero.
Then the sets E_k obtained by taking intersection $A_i \cap B_j$ form a finite disjoint collection of measurable sets.
Hence φ and ψ can be written as

$$\varphi = \sum_{k=1}^{N} a_k \chi_{E_k} ,$$

$$\psi = \sum_{k=1}^{N} b_k \chi_{E_k} .$$

Therefore ,

$$a\varphi + b\psi = \sum_{k=1}^{N} (aa_k + bb_k\}\chi_{E_k}$$

$$= a\sum_{k=1}^{N} a_k \chi_{E_k} + b\sum_{k=1}^{N} b_k \chi_{E_k} .$$

Therefore ,

$$\int (a\varphi + b\chi) = a\sum_{k=1}^{N} a_k mE_k + b\sum_{k=1}^{N} b_k mE_k .$$

$$= a\int \varphi + b\int \chi .$$

Hence ,

$$\int (a\varphi + b\chi) = a\int \varphi + b\int \chi . \qquad \ldots(1)$$

If $\varphi \quad \psi$ a.e.,
then $\varphi - \psi \quad 0$ a.e.
Since the integral of a simple function is always greater than or equal to zero,

$$\int (\varphi - \chi) \quad 0.$$

From (1)

$$\int (\varphi - \chi) = \int \varphi - \int \chi$$

Therefore,

$$\int \varphi - \int \chi \quad 0.$$

Hence ,

$$\int \varphi \quad \int \chi .$$

Hence the theorem.

Remark:

From theorem 18.2, if $\varphi = \sum_{i=1}^{n} a_i \chi_{E_u}$, then

$$\int \varphi = \sum_{i=1}^{n} a_i mE_i .$$

Hence the restriction of theorem18.1 that the sets E_i be distinct is unnecessary.

Theorem18.3

Let f be defined and bounded on a measurable set E with mE finite. In order that

$$\inf_{f \le \psi} \int_E \psi(x)dx = \sup_{f \ge \varphi} \int_E \varphi(x)dx$$

for all simple functions φ and ψ, it is necessary and sufficient that f be measurable.

Proof

Let f be bounded by M.

i.e $|f(x)|$ M, for all x in E.

Assume that f is measurable.

Then the sets

$$E_k = \{x \,/\, \frac{kM}{n} \geq f(x) > \frac{(k-1)M}{n}\}, \; -n \quad k \quad n,$$

are measurable, disjoint and

$$E = \bigcup_{k=-n}^{n} E_k .$$

Therefore $mE = \sum_{k=-n}^{n} mE_k$.

For n=1,2,3,….,

define the simple functions φ_n and ψ_n by

$$\psi_n(x) = \frac{M}{n}\sum_{k=-n}^{n} k\chi_{E_k}(x) \text{ and}$$

$$\varphi_n(x) = \frac{M}{n}\sum_{k=-n}^{n} (k-1)\chi_{E_k}(x)$$

Hence $\varphi_n(x) \quad f(x) \quad \psi_n(x)$, n=1,2,3,…

Therefore

$$\inf_{f\leq\psi}\int_E \psi(x)dx \quad \int_E \psi_n(x)dx = \frac{M}{n}\sum_{k=-n}^{n} kmE_k \text{ and}$$

$$\sup_{f\geq\varphi}\int_E \varphi(x)dx \quad \int_E \varphi_n(x)dx = \frac{M}{n}\sum_{k=-n}^{n} (k-1)mE_k .$$

Hence $0 \quad \inf_{f\leq\psi}\int_E \psi(x)dx - \sup_{f\geq\varphi}\int_E \varphi(x)dx$

$$\frac{M}{n}\sum_{k=-n}^{n} kmE_k - \frac{M}{n}\sum_{k=-n}^{n} (k-1)mE_k$$

$$= \frac{M}{n}\sum_{k=-n}^{n} mE_k$$

$$= \frac{M}{n} mE .$$

Since lim $_{n›}$ $\frac{M}{n} mE$ =0, we get that

$$\inf_{f\leq\psi}\int_E \psi(x)dx = \sup_{f\geq\varphi}\int_E \varphi(x)dx .$$

Hence the condition is sufficient.

To prove the necessary part,

Assume that $\inf_{f\leq\psi}\int_E \psi(x)dx = \sup_{f\geq\varphi}\int_E \varphi(x)dx$.

Then given n ,

there are simple functions φ_n and ψ_n such that

$$\varphi_n(x) \quad f(x) \quad \psi_n(x),\ n=1,2,3,\ldots \qquad \ldots(1).$$

and

$$\int_E \psi_n(x)dx - \int_E \varphi_n(x)dx < 1/n.$$

Since ψ_n are measurable,

$\psi^* = \inf \psi_n$ is measurable. [by theorem 17.4]

Similarly since φ_n are measurable,

$\varphi^* = \sup \varphi_n$ are measurable.

Also from (1),

$$\varphi^*(x) \quad f(x) \quad \psi^*(x).$$

Consider the set $\Delta = \{x / \varphi^*(x) < \psi^*(x)\}$

$= \bigcup \Delta_\upsilon$

where $\Delta_\upsilon = \{x/\varphi^*(x) < \psi^*(x) - \frac{1}{\upsilon}\} \subset \{x/\varphi_n(x) < \psi_n(x) - \frac{1}{\upsilon}\}$.

From (1), the latter set has measure less than ν/n.

Since n is arbitrary, $m\Delta_\nu = 0$.

Hence $m\Delta = 0$.

Hence $\varphi^* = \psi^*$ except on a set of measure zero.

Hence $\varphi^* = f$ except on a set of measure zero.

Hence f is measurable. [by theorem 17.5]

Hence the condition is also necessary.

Hence the theorem.

18.3 The Lebesgue integral of bounded functions over a set of finite measure

Definition

If f is a bounded measurable function defined on a measurable set E with mE finite, we define the Lebesgue integral of f over E by

$$\int_E f(x)dx = \inf \int_E \psi(x)dx \quad \text{for all simple functions } \psi \quad f.$$

Remark:

1.If f vanishes outside a set E of finite measure,

we write $\int f$ for $\int_E f$.

2. the following theorem shows that Lebesgue integral is the generalization of the Riemann integral.

Definition

A step function is a function ψ which has the form

$\psi(x) = c_i$, if $x_{i-1} < x < x_i$,

for some subdivision $\{x_0, x_1, \ldots x_n\}$ of [a, b] and some constants c_i.

Therefore by the definition of integral,

$$\int_a^b \psi(x)dx = \sum_{i=1}^{n} c_i(x_i - x_{i-1}).$$

Hence upper Riemann integral of a function f can be defined as

$$R\overline{\int_a^b} f(x)dx = \inf \int_a^b \psi(x)dx,$$

where the infinimum is taken over all step functions $\psi(x)\quad f(x)$.

And the lower Riemann integral of f is defined as

$$\mathrm{R}\underline{\int_a^b} f(x)dx = \sup \int_a^b \varphi(x)dx \ ,$$

where the supremum is taken over all step functions $\varphi(x)\quad f(x)$.

Also we know that f is Riemann integrable if and only if

$$\mathrm{R}\overline{\int_a^b} f(x)dx \ = \quad \mathrm{R}\underline{\int_a^b} f(x)dx$$

and the common value is denoted by $\mathrm{R}\int_a^b f(x)dx$.

Theorem18.4

Let f be a bounded function defined on [a, b].

If f is Riemann integrable on [a, b], then it is measurable and

$$\mathrm{R}\int_a^b f(x)dx \ = \int_a^b f(x)dx \, .$$

Proof

Since every step function is a simple function,

$$\begin{aligned}\mathrm{R}\underline{\int_a^b} f(x)dx \quad & \sup_{f\geq\varphi}\int_E \varphi(x)dx \\ & \inf_{f\leq\psi}\int_E \psi(x)dx \\ & \mathrm{R}\overline{\int_a^b} f(x)dx \, .\end{aligned}$$

Since f is Riemann integrable,

the inequalities are all equalities.

Hence f is measurable and

$$\mathrm{R}\int_a^b f(x)dx \ = \int_a^b f(x)dx \, .$$

Hence the theorem.

Theorem18.5

If f and g are bounded measurable functions defined on a set E of finite measure, then

(i) $\int_E (af + bg) = a\int_E f \ + b\int_E g$.

(ii) If f = g a.e., then $\int_E f \ = \int_E g$.

(iii) If f g a.e., then $\int_E f \quad \int_E g$.

Hence $|\int_E f| \quad \int_E |f|$.

(iv) If A f(x) B, then AmE $\int_E f$ BmE.

(v) If A and B are disjoint measurable sets of finite measure, then

$$\int_{A\cup B} f \ = \int_A f \ + \int_B f \, .$$

Proof

(i) To prove that $\int_E (af+bg) = a\int_E f + b\int_E g$.

If ψ is a simple function,
then aψ is also a simple function(if a 0).

Hence for a >0 ,

$$\int_E af = \inf_{\psi \geq f} \int_E a\psi$$

$$= a \inf_{\psi \geq f} \int_E \psi$$

$$= a \int_E f \, .$$

If a <0, then

$$\int_E af = \sup_{\varphi \leq f} \int_E a\varphi$$

$$= a \inf_{\psi \geq f} \int_E \psi$$

$$= a \int_E f \, . \qquad \text{[by theorem18.3]}$$

Hence if a 0,

$$\int_E af = a \int_E f \, . \qquad \ldots(1)$$

If ψ₁ and ψ₂ are simple functions such that f ψ₁ and g ψ₂,
then ψ₁ + ψ₂ is also a simple function such that f + g ψ₁ + ψ₂.

Hence

$$\int_E f+g = \inf \int_E (\psi_1 + \psi_2)$$

$$\quad \int_E (\psi_1 + \psi_2)$$

$$= \int_E \psi_1 + \int_E \psi_2 \, , \qquad \text{for all } \psi_1 \quad f \text{ and } \psi_2 \quad f.$$

Hence,

$$\int_E f+g \quad \inf \int_E \psi_1 + \inf \int_E \psi_2 \, .$$

i.e $$\int_E f+g \quad \int_E f + \int_E g \, . \qquad \ldots(2)$$

On the other hand ,
if φ₁ and φ₂ are simple functions such that φ₁ f and φ₂ f,
then φ₁+ φ₂ is also a simple function such that φ₁ + φ₂ f.

Hence

$$\int_E f+g = \sup \int_E (\varphi_1 + \varphi_2)$$

$$\quad \int_E \varphi_1 + \int_E \varphi_2 \, , \qquad \text{for all } \varphi_1 \quad f \text{ and } \varphi_2 \quad f.$$

Hence,

$$\int_E f+g \quad \sup \int_E \varphi_1 + \sup \int_E \varphi_2 \, .$$

Hence ,

$$\int_E f+g \quad \int_E f + \int_E g \qquad \ldots(3)$$

From(2) and (3), we get

$$\int_E f+g = \int_E f + \int_E g \qquad \ldots(4)$$

Therefore from (1) and (4), we get

$$\int_E (af + bg) = \int_E af + \int_E bg$$

$$= a\int_E f + b\int_E g .$$

Hence (i) is proved.

(ii) To prove that, if f = g a.e., then $\int_E f = \int_E g$.

Since f = g a.e.,

f – g = 0 a.e.

Hence,

if ψ f – g , then ψ 0 a.e.

Hence ,

$$\int_E \psi \quad 0, \qquad \text{for all } \psi \quad f - g .$$

Therefore,

$$\inf \int_E \psi \quad 0.$$

Hence,

$$\int_E (f - g) \quad 0.$$

From (i), $\int_E (f - g) = \int_E f - \int_E g$

Hence ,

$$\int_E f - \int_E g \quad 0.$$

Hence ,

$$\int_E f \qquad \int_E g \qquad \text{…(5).}$$

Similarly,

if φ f – g ,

then φ 0 a.e

Hence ,

$$\int_E \varphi \quad 0 \qquad \text{for all } \varphi \quad f - g.$$

Hence,

$$\sup \int_E \varphi \quad 0.$$

Hence ,

$$\int_E (f - g) \quad 0.$$

Hence,

$$\int_E f - \int_E g \quad 0.$$

Hence ,

$$\int_E f \qquad \int_E g \qquad \text{…(6)}$$

From (5) and (6), we get

$$\int_E f = \int_E g .$$

Hence (ii) is proved.

(iii) To prove that, if f $\quad$ g a.e., then $\int_E f \quad \int_E g$.

If f $\quad$ g a.e.,
then f –g $\quad$ 0 a.e.

Hence,

if φ $\quad$ f –g ,
then φ $\quad$ 0 a.e

Hence ,

$$\int_E \varphi \quad 0 \qquad\qquad \text{for all } \varphi \quad f-g.$$

Hence,

$$\sup \int_E \varphi \quad 0.$$

Hence ,

$$\int_E (f-g) \quad 0.$$

Hence ,

$$\int_E f \ - \int_E g \quad 0.$$

Hence,

$$\int_E f \qquad \int_E g .$$

Since $\quad$ f $\quad$ |f|,
from this inequality we get ,

$$\int_E f \qquad \int_E |\, f\,|. \qquad \ldots(7)$$

Similarly since –f $\quad$ |f|,

$$-\int_E f \qquad \int_E |\, f\,|. \qquad \ldots(8)$$

From (7) and (8),

$$|\int_E f\,| \quad \int_E |\, f\,|.$$

Hence (iii) is proved.

(iv) To prove that if A $\quad$ f(x) $\quad$ B, then AmE $\quad \int_E f \quad$ BmE.

From(iii),

if A $\quad$ f(x) , $\qquad \int_E A \qquad \int_E f$. $\qquad$ …(9)

Since A is a constant ,

$$\int_E A = A \int_E 1$$
$$= \text{AmE}.$$

Therefore from (9),

$$\text{AmE.} \quad \int_E f . \qquad \ldots(10)$$

Similarly if f(x) $\quad$ B ,

$$\text{then } \int_E f \qquad \int_E B$$
$$= \text{BmE} \qquad \ldots(11)$$

From (10) and (11), we get that

if A $\quad$ f(x) $\quad$ B, $\qquad$ then AmE $\quad \int_E f \quad$ BmE.

Hence (iv) is proved.

(v) To prove that if A and B are disjoint measurable sets of finite measure, then

$$\int_{A\cup B} f = \int_A f + \int_B f .$$

Since A and B are measurable sets,

A $\cup$ B is also a measurable set.

Since A and B are also disjoint,

$$\chi_{A\cup B} = \chi_A + \chi_B .$$

Therefore

$$\begin{aligned}\int_{A\cup B} f &= \int f \cdot \chi_{A\cup B}\\ &= \int f \cdot (\chi_A + \chi_B)\\ &= \int f \cdot \chi_A + \int f \cdot \chi_B\\ &= \int_A f + \int_B f .\end{aligned}$$

Hence (v) is proved.
Hence the theorem.

Theorem18.6 Bounded convergence theorem

Let { f_n } be a sequence of measurable functions defined on a set E of finite measure, and suppose that there is a real number M such that

$|f_n(x)|$ M for all n and all x.

If $f(x) = \lim f_n(x)$ for each x in E, then

$$\int_E f = \lim \int_E f_n .$$

Proof

Let { f_n } be a sequence of measurable functions defined on a set E of finite measure, and suppose that there is a real number M such that

$|f_n(x)|$ M , for all n and all x. …(1)

Let $f(x) = \lim f_n(x)$, for each x in E. …(2)

Hence by Littlewood's third principle,

for given $\varepsilon > 0$ and $\delta = \varepsilon/4M$,

there is an integer N and a measurable set $A \subset E$ such that

for n N and $x \in E - A$,

$|f_n(x) - f(x)| < \varepsilon/2mE$. …(3)

From (1), for all x in A and for all n,

$|f_n(x) - f(x)|$ $|f_n(x)| + |f(x)|$

M +M=2M …(4)

Hence

$$\begin{aligned}\left|\int_E f_n - \int_E f\right| &= \left|\int_E (f_n - f)\right|\\ &\ \int_E | f_n - f |\\ &= \int_{E-A} | f_n - f | + \int_A | f_n - f |\end{aligned}$$

< E-A(ε/2mE) + A(2M) , for n N [From(3) and (4)]

= (ε/2mE)m(E –A) +(2M)mA

< (ε/2mE)mE +(2M)(ε/4M)

$= \varepsilon/2 + \varepsilon/2$

$= \varepsilon,$ for n N .

Hence ,

$$\int_E f_n › \int_E f \ .$$

Check your progress

Prove that a bounded function f on [a, b] is Riemann integrable if and only if the set of points at which f is discontinuous has measure zero.

18.4 Let us sum up

In this lesson we have studied

- The definition of a simple function
- The integral of a simple function
- Lebesgue integral of a function
- Properties of the integral and
- Bounded convergence theorem.

18.5 Lesson End Activities

1. Prove that the sum and product of two simple functions are simple.
2. set f be a nonnegatve measurable function. Show that ∫ f = 0 => f =0 a.e

18.6 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

Lesson 19 The integral of a non-negative function

19.0 Introduction
19.1 Aims and Objectives
19.2 The integral of a non-negative function
19.3 Let us sum up.
19.4 Lesson End Activities
19.5 References

19.0 Introduction

In this lesson we are going to study about the definition and the properties of the integral of non-negative functions and some important theorems.

19.1 Aims and Objectives

After studying this lesson, you would know

- How to find the integral of a non-negative function?
- Properties of the integral of non-negative functions.
- Fatou's Lemma
- Monotone convergence theorem and
- Definition of Integrable function over a measurable set .

19.2 The integral of a non-negative function

Definition

If f is a non-negative measurable function defined on a measurable set E, the integral of f is defined as

$$\int_E f = \sup_{h \le f} \int_E h ,$$

where h is a bounded function such that

m{x / h(x) 0} is finite.

Theorem19.1

If f and g are non-negative measurable functions, then:

(i) $\int_E cf = c \int_E f$, c > 0.

(ii) $\int_E f + g = \int_E f + \int_E g$.

(iii) If f g a.e., then $\int_E f \quad \int_E g$.

Proof

(i) To prove that $\int_E cf = c \int_E f$, c > 0.

For c >0, consider

$$\begin{aligned} \int_E cf &= \sup_{h \le f} \int_E ch \\ &= c \sup_{h \le f} \int_E h \\ &= c \int_E f . \end{aligned}$$

Hence (i) is proved.

(ii) To prove that $\int_E f+g = \int_E f + \int_E g$.

If h and k are bounded measurable functions such that

h(x) f(x) and

k(x) g(x),

then h+k is also a bounded measurable function such that

h(x) +k(x) f(x) + g(x).

Hence ,

$$\int_E h + \int_E k = \int_E h+k$$

$$\text{sup } \int_E h+k$$

$$= \int_E f+g \text{ .}$$

Hence ,

$$\text{sup} \int_E h \text{ +sup } \int_E k \qquad \int_E f+g$$

Hence $\int_E f + \int_E g \quad \int_E f+g$ …(1)

On the other hand,
let l be a bounded measurable function which vanishes outside a set of measure zero and which is not greater than f + g.
Define the functions h and k by setting

h(x) = min(f(x), l(x))

and k(x) = l(x) –h(x).

Then h(x) f(x) and

k(x) g(x).

Also h and k are bounded measurable functions by the bound for l and vanish where l vanishes.

Hence ,

$$_E\, l = {}_E\, h + {}_E\, k$$

$$\int_E f + \int_E g$$

Hence,

$$\sup_{l \le f+g} \int_E l \qquad \int_E f + \int_E g \text{ .}$$

Hence,

$$\int_E f+g \qquad \int_E f + \int_E g \text{ .} \qquad \text{…(2)}$$

From (1) and (2),

$$\int_E f+g = \int_E f + \int_E g \text{ .}$$

Hence (ii) is proved.

(iii) To prove that if f g a.e., then $\int_E f \quad \int_E g$.

If f g a.e.,

then f –g 0 a.e.

Therefore if h is a bounded measurable function such that

h(x) (f –g)(x),

then h 0 a.e.

Hence,

$$\int_E h \quad 0.$$

Therefore,

$$\sup \int_E h \quad 0.$$

Hence ,

$$\int_E f - g \quad 0.$$

Hence ,

$$\int_E f - \int_E g \quad 0.$$

Hence,

$$\int_E f \quad \int_E g\,.$$

Hence (iii) is proved.
Hence the theorem.

Theorem19.2 Fatou's Lemma:
If { f_n } is a sequence of nonnegative measurable functions and $f_n(x)›f(x)$ almost everywhere on a set E, then

$$\int_E f \quad \underline{\lim} \int_E f_n\,.$$

Proof

Without loss of generality, assume that $f_n(x)›f(x)$ everywhere, since integrals over a set of measure zero are zero.
Let h be a bounded measurable function which is not greater than f and which vanishes outside a set E' of finite measure.
For n=1,2,,3,…
define a function h_n by setting

$$h_n(x) = \min \{ h(x), f_n(x)\}.$$

Then h_n is bounded by the bound for h and vanishes outside E'.
Also $h_n(x) › h(x)$ for each x in E'.
Hence by Bounded convergence theorem,

$$\int_E h = \int_{E'} h = \lim \int_{E'} h_n$$
$$\underline{\lim} \int_E f_n\,.$$

This is true for all bounded measurable functions h which is not greater than f and which vanishes outside a set of finite measure.
Hence taking supremum over h,

$$\sup \int_E h \quad \underline{\lim} \int_E f_n\,.$$

Hence $$\int_E f \quad \underline{\lim} \int_E f_n\,.$$

Hence the theorem.

Theorem 19.3 Monotone convergence theorem:

Let {f_n }be an increasing sequence of nonnegative measurable functions, and let f = lim f_n a.e.

Then $$\int f = \lim \int f_n\,.$$

Proof

By theorem 19.2,

$$\int f \quad \underline{\lim} \int f_n\,. \qquad \ldots(1)$$

Since$\{f_n\}$ is an increasing sequence nonnegative measurable functions, and $f = \lim f_n$ a.e.,

for each n, $f_n \quad f$.

Hence $\int f_n \quad \int f$.

Therefore,

$$\overline{\lim}\int f_n \quad \int f \qquad \text{…(2)}$$

From (1) and (2),

$$\overline{\lim}\int f_n \quad \int f \quad \underline{\lim}\int f_n \ .$$

But we know that, it is always true that,

$$\underline{\lim}\int f_n \quad \overline{\lim}\int f_n \qquad \text{..(3)}$$

From (2) and (3),

$$\underline{\lim}\int f_n = \overline{\lim}\int f_n = \int f$$

Hence,

$$\int f = \lim \int f_n \ .$$

Hence the theorem.

Theorem 19.4

Let $\{u_n\}$ be s sequence of nonnegative measurable functions, and

Let $f = \sum_{n=1}^{\infty} u_n$.

Then $\int f = \sum_{n=1}^{\infty} \int u_n$.

Proof

Let $s_n = \sum_{k=1}^{n} u_k$.

Then $\{s_n\}$ is also a sequence of nonnegative measurable functions such that

s_n› f .

i.e $f = \lim s_n$.

Hence by theorem 19.3,

$$\int f = \lim \int s_n$$

$$=\lim \int \sum_{k=1}^{n} u_k$$

$$=\lim \sum_{k=1}^{n} \int u_k$$

$$= \sum_{n=1}^{\infty} \int u_n \ .$$

Hence,

$$\int f = \sum_{n=1}^{\infty} \int u_n \ .$$

Hence the theorem.

Theorem19.5

Let f be a nonnegative function and { E_i } b a disjoint sequence of measurable sets. Let E = $\cup E_i$. Then

$$\int_E f = \sum \int_{E_i} f \, .$$

Proof

Since E = $\cup E_i$,

$$\chi_E = \sum \chi_{E_i}$$

Let $u_i = f \cdot \chi_{E_i}$.

Then

$$\begin{aligned} f \cdot \chi_E &= f \cdot \sum \chi_{E_i} \\ &= \sum f \cdot \chi_{E_i} \\ &= u_i. \end{aligned}$$

Hence by theorem 19.4,

$$\begin{aligned} \int_E f &= \int f \cdot \chi_E \\ &= \int \sum u_i \\ &= \sum \int u_i \\ &= \sum \int f \cdot \chi_{E_i} \\ &= \sum \int_{E_i} f \, . \end{aligned}$$

Hence the theorem.

Definition

A nonnegative measurable function f is called integrable over the measurable set E if

$$\int_E f < \ .$$

Theorem19.6

Let f and g be two nonnegative measurable functions.
If f is integrable over E and g(x) < f(x) on E,
then g is also integrable on E, and

$$\int_E f - g = \int_E f - \int_E g \, .$$

Proof

Since f = (f –g) +g, By theorem19.1,

$$\int_E f = \int_E f - g + \int_E g \, .$$

Since the left side $\int_E f$ is finite,

the terms on the right side is also finite.

Hence,

$$\int_E g \text{ is finite .}$$

Hence g is integrable and

$$\int_E f-g = \int_E f - \int_E g .$$

Hence the theorem.

Theorem 19.7

Let f be a nonnegative function which is integrable over a set E.
Then given ε >0,
there is a δ >0 such that for every set A ⊂ E with mA < δ, we have

$$\int_A f < \varepsilon.$$

Proof

If f is bounded, then the theorem is trivial.

Let $f_n(x) = f(x)$ if f(x) n

and $f_n(x) = n$ otherwise.

Then each f_n is bounded and
f_n converges to f at each point.
Hence by monotone convergence theorem,

$$\int f = \lim \int f_n .$$

Hence there is an N such that

$$\int_E f_N > \int_E f - \varepsilon/2.$$

Hence,

$$\int_E (f - f_N) = \int_E f - \int_E f_N$$
$$< \varepsilon/2.$$

Choose δ < ε/2N.
Now if A ⊂ E with mA < δ,

Then $\int_A f = \int_A f - f_N + \int_A f_N$

$< \int_E (f - f_N) + NmA$

$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$

Hence given ε >0 there is a δ >0 such that for every set A ⊂ E with mA < δ, we have

$$\int_A f < \varepsilon.$$

Hence the theorem.

Check your progress

1.Show that w may have strict inequality in Fatou's Lemma.
Hint: consider the sequence $\{f_n\}$ defined by $f_n(x) = 1$ if n x <n+1,with $f_n(x)=0$ otherwise.
2. Show that the monotone convergence theorem need not hold for decreasing sequence of functions.
Hint :Let $f_n(x)=0$, if x< n, $f_n(x) = 1$ for x n .

19.3 Let us sum up

In this lesson we have studied

- The definition of integral of nonnegative measurable function
- The definition of integrable function
- Properties of the integrals of nonnegative measurable function
- Fatou's lemma
- Monotone convergence theorem and

- a property of a nonnegative integrable function .

19.4 Lesson End Activities

1. Show that if f and g are measurable & y |f| ≤|g| a.e., and if g is ntergrable, then prove that f is intergrable

19.5 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

LESSON – 20

THE GENERAL LEBESGUE INTEGRAL AND CONVERGENCE IN MEASURE.

Contents

20.0 Introduction

In this lesson , you are going to study about the general Lebesgue integral, some of its properties, convergence in measure and theorems related to them.

20.1 Aims and objectives

After studying this lesson, you would know

- What is the positive part of a function?
- What is the negative part of a function?
- Definition of General Lebesgue integral of a measurable function
- Properties of Lebesgue integral .
- Lebesgue convergence theorem
- Generalization of Lebesgue convergence theorem
- Definition of convergence in measure of a sequence of measurable functions and
- Every sequence of measurable sequence that converges in measure contains a subsequence that converges almost everywhere.

20.2 The General Lebesgue integral

Definition

The positive part of a function f is $f^+ = f \vee 0$

i.e $f^+(x) = \max\{ f(x), 0\}$

The negative part of a function is $f^- = f \wedge 0$.

i.e $f^-(x) = \min \{f(x), 0\}$

Hence $f = f^+ - f^-$.

And $|f| = f^+ + f^-$

Definition

A measurable function f is said to be integrable over E if f^+ and f^- are both integrable over E.

Then the integral of f is defined as

$$\int_E f = \int_E f^+ - \int_E f^- .$$

Theorem 20.1

Let f and g are integrable over E. Then

(i) The function cf is integrable over E, and

$$\int_E cf = c\int_E f .$$

(ii) The function f+g is integrable over E, and

$$\int_E f + g = \int_E f + \int_E g .$$

(iii) If $f \quad g$ a.e., then $\int_E f \quad \int_E g$.

(iv) If A and B are disjoint measurable sets contained in E, then

$$\int_{A\cup B} f = \int_A f + \int_B f$$

Proof

(i) Since f is integrable over E ,
both f^+ and f^- are integrable over E and
the integral of f is given by

$$\int_E f = \int_E f^+ - \int_E f^- .$$

Hence,

both cf^+ and cf^- are integrable over E, and

hence,

$cf = cf^+ - cf^-$ are integrable over E and

$$\int_E cf = \int_E cf^+ - \int_E cf^- .$$

$$=c \int_E f^+ -c \int_E f^- \qquad \text{[by theorem 19.1]}$$

$$=c[\int_E f^+ - \int_E f^-]$$

$$=c\int_E f .$$

Hence (i) is proved.

(ii) Suppose if f_1 and f_2 are nonnegative integrable functions with $f= f_1 - f_2,$

Then $f^+ - f^- = f_1 - f_2.$

Hence ,

$$f^+ + f_2 = f^- + f_1.$$

Hence by theorem19.1,

$$f^+ + f_2 = f^- - f_1.$$

Therefore,

$$f = f^+ - f^-$$

$$= f_1 - f_2. \qquad \ldots(1)$$

Since f and g are measurable,

f^+ , f^-, g^+, g^- are measurable.

Hence,

$f^+ + g^+$, $f^- + g^-$ are also measurable.

And $f + g = (f^+ + g^+) -(f^- + g^-)$.

Hence by(1) and theorem19.1,

$$(f + g) = (f^+ + g^+) - (f^- + g^-)$$

$$= f^+ + g^+ - f^- - g^-$$

$$= (f^+ - f^-) +(g^+ - g^-)$$

$$= f + g.$$

Hence (ii) is proved.

(iii) Since f g a.e.,

$$f^{+} - f^{-} \quad g^{+} - g^{-} \text{ a.e.,}$$

hence ,

$$f^{+} + g^{-} \quad g^{+} + f^{-} \text{ a.e.,}$$

Hence by using the results of theorem19.1

$$(f^{+} + g^{-}) \qquad (g^{+} + f^{-}).$$

Hence

$$f^{+} + \quad g^{-} \qquad g^{+} + \quad f^{-}.$$

Hence ,

$$f^{+} - \quad f^{-} \qquad g^{+} - \quad g^{-}$$

Hence,

$$f \qquad g.$$

Hence (iii) is proved.

(iv) Consider $\int_{A\cup B} f = \int f \cdot \chi_{A\cup B}$

$$= \int f \cdot (\chi_A + \chi_B)$$

$$= \int f \cdot \chi_A + \int f \cdot \chi_B$$

$$= \int_A f + \int_B f .$$

Hence (iv) is proved.
Hence the theorem.

Theorem 20.2 Lebesgue Convergence theorem.

Let g be integrable over E and let $\{ f_n \}$ be a sequence of measurable functions such that $|f_n|$ g on E and
for almost all x in E we have $f(x) = \lim f_n(x)$. Then

$$\int_E f = \lim \int_E f_{n} .$$

Proof

Since $|f_n|$ g on E,

$g - f_n$ is nonnegative and hence by Fatou's Lemma,

$$\int_E (g - f) \qquad \underline{\lim} \int_E (g - f_n) . \qquad \ldots(1)$$

Since $f(x) = \lim f_n(x)$ a.e. on E and
$|f_n|$ g on E,

$$|f| \quad g \text{ on E.}$$

Hence since g is integrable,
f is also integrable.
Therefore,

$$\int_E (g - f) = \int_E g - \int_E f \qquad \ldots(2)$$

Also,

$$\underline{\lim} \int_E (g - f_n) = \int_E g - \overline{\lim} \int_E f_n \qquad \ldots(3)$$

Substituting (2) and (3) in (1),we get

$$\int_E g - \int_E f \qquad \int_E g - \overline{\lim} \int_E f_n$$

Hence

$$\int_E f \qquad \overline{\lim}\int_E f_n \, . \qquad \ldots(4).$$

Similarly by considering g +f_n, we get

$$\int_E f \qquad \underline{\lim}\int_E f_n \qquad \ldots(5)$$

From (4) and (5), we get

$$\overline{\lim}\int_E f_n \qquad \int_E f \qquad \underline{\lim}\int_E f_n \, . \qquad \ldots(6)$$

But it is always true that

$$\underline{\lim}\int_E f_n \qquad \overline{\lim}\int_E f_n \qquad \ldots(7)$$

From (6) and (7),

$$\int_E f = \lim \int_E f_{\ n} \, .$$

Hence the theorem.

Remark:

If we replace g by g_n's, we get the following generalization of the Lebesgue Convergence theorem.

Theorem 20.3

Let { g_n} be a sequence of integrable functions which converges a.e to an integrable function g. Let {f_n} be a sequence of measurable functions such that

$|f_n|$ g_n and

{ f_n} converges to f a.e.

If $\int g = \lim \int g_n$,

then $\int f = \lim \int f_n$.

Check your progress

1. Show that if f is integrable over E, then so is |f| and
   $$|\int_E f| \le \int_E |f| \, .$$

Does the integrability of |f| imply that of f?.

2. Show that under the hypotheses of theorem20.3, we have
   $$\int |f_n - f| \to 0 \, .$$

3.Let {f_n } be a sequence of integrable functions such that f_n›f a.e with f integrable. Then $\int |f_n - f| \to 0$ if and only if $\int |f_n| \to \int |f|$.

20.3 Convergence in measure

Definition

A sequence { f_n} of measurable functions is said to converge to f in measure if, given ε >0, there is an N such that for all n N we have

m{ x / |f(x) –f_n(x)| ε } < ε.

Remark:

From this definition and littlewood's third principle, it is clear that,

If { f_n} is a sequence of measurable functions defined on a measurable set E of finite measure and f_n›f a.e, then { f_n } converges to f in measure.

Example

Construct the sequence $\{ f_n \}$ as follows:
Let $\quad n=k+2^{\nu}$, $0 \quad k < 2^{\nu}$, and
Set $\quad f_n(x) = 1$ if $x \in [k2^{-\nu}, (k+1)\, 2^{-\nu}]$
And $\quad f_n(x) = 0$ otherwise.
Then $\quad m\{ x/\, |f_n(x)| > \varepsilon \} = 2^{-\nu} \quad 2/n$ $\qquad$ [since $2^{\nu} \quad n<2^{\nu+1}$]
Hence $f_n ›\ 0$ in meaure.

Remark:
Note that the sequence $\{f_n(x)\}$ has the value 1 for arbitrarily large values of n.
Hence $\{f_n(x)\}$ does not converge for any x in $[0,1]$.

Theorem 20.4

Let $\{f_n\}$ be a sequence of measurable functions that converges in measure to f. Then there is a subsequence $\{ f_{n_k} \}$ that converges to f almost everywhere.

Proof

Since $\{f_n\}$ is a sequence of measurable functions that converges in measure to f,
Given ν,
there is an integer n_ν such that for all $n \quad n_\nu$,

$$m\{ x / |f(x) - f_n(x)| \quad 2^{-\nu} \} < 2^{-\nu} . \qquad ..(1)$$

Let $\quad E_\nu = \{ x / | f_{n_\upsilon}(x) - f(x)| \quad 2^{-\nu} \}$.

Therefore ,

if $\quad x \notin \bigcup_{\upsilon=k}^{\infty} E_\upsilon$,

then $\quad | f_{n_\upsilon}(x) - f(x)| < 2^{-\nu}$ for $\nu\ k$.

Therefore ,

$$f_{n_\upsilon}(x) › f(x).$$

Hence $f_{n_\upsilon}(x) › f(x)$ for any $x \notin A = \bigcap_{k=1}^{\infty} \bigcup_{\upsilon=k}^{\infty} E_\upsilon$.

But

$$mA \quad m\Big[\bigcup_{\upsilon=k}^{\infty} E_\upsilon \Big]$$

$$\sum_{\upsilon=k}^{\infty} mE_\upsilon$$

$$=2^{-k+1}.$$

Hence $mA = 0$

Hence there there is a subsequence $\{ f_{n_k} \}$ that converges to f almost everywhere.

Theorem 20.5

Let $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E of finite measure.
Then $\{f_n\}$ converges to f in measure if and only if every subsequence of $\{f_n\}$ has in turn a subsequence that converges almost everywhere to f.

Theorem20.6

Fatou's lemma and the monotone and Lebesgue Convergence theorem remain valid if 'convergence a.e.' is replaced by 'convergence in measure'.

Check your progress.

1. Prove theorem 20.5
2. Prove theorem 20.6

20.4 Let us sum up

In this lesson , we have studied

- Definition of General Lebesgue integral of a measurable function
- Properties of Lebesgue integral .
- Lebesgue convergence theorem
- Generalization of Lebesgue convergence theorem
- Definition of convergence in measure of a sequence of measurable functions and
- Every sequence of measurable sequence that converges in measure contains a subsequence that converges almost everywhere.

20.5 Lesson End Activities

1. Show that if f is integrable over E, then $|f|$ is also integrable over E. further

$$\left|\int_E f\right| \leq \int_E |f|$$

is the converse true?

20.6 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

Unit 5 Differentiation and Integration

Lesson 21 Differentiation of monotone functions

21.0 Introduction
21.1 Aims and Objectives
21.2 Differentiation of monotone functions
21.3 Let us sum up
21.4 Lesson End Activities
21.5 References

21.0 Introduction

In this lesson, we are going to study about vitali covering of a set and the differentiation of monotone functions.

21.1 Aims and Objectives

After studying this lesson, you would know,

- The vitali covering of a set
- A monotone function on an interval is differentiable and
- The derivative of the monotone function is measurable.

21.2 Differentiation of monotone functions

Definition: Vitali covering

Let J be a collection of intervals. We say that J covers a set E in the sense of Vitali, if for each $\varepsilon > 0$ and any x in E, there is an interval $I \in J$

such that $x \in I$ and $l(I) < \varepsilon$.

The intervals may be open, closed or half open, but degenerate intervals consisting of only one point is not allowed.

Theorem21.1 Vitali Lemma

Let E be a set of finite outer measure and J a collection of intervals that cover in the sense of vitali, Then, given $\varepsilon > 0$, there is a finite disjoint collection $\{ I_1, I_2, \ldots I_N\}$ of intervals in J such that

$$m^*\left[E - \bigcup_{n=1}^{N} I_n\right] < \varepsilon.$$

Proof

Without loss of generality, assume that the intervals in J are closed, for otherwise , we can replace each interval by its closure.

Since the measure of the set of end-points of the intervals $\{ I_1, I_2, \ldots I_N\}$ has measure zero, our assumption would not affect the result.

Let O be an open set of finite measure containing E.

Since J is a vitali covering of E and $E \subset O$,

we may assume that each I of J is contained in O.

Construct a sequence $\{ I_n\}$ of disjoint intervals of J by induction as follows.

1. Let I_1 be any interval in J.
2. Suppose $I_1, I_2, \ldots I_n$ are already been chosen.
3. Let k_n be the supremum of the lengths of the intervals of J that do not meet any of the intervals $I_1, I_2, \ldots I_n$.
4. Since each $I_n \subset O$, k_n mO < .

5. Unless E $\subset \bigcup_{i=1}^{n} I_i$, we can find I_{n+1} in J with $l(I_{n+1})$ > (½)k_n and

I_{n+1} disjoint from $I_1, I_2, \ldots I_n$.

Hence we have a sequence { I_n } of disjoint intervals of J.

Since each $I_n \subset O$,

$\bigcup I_n \subset O$.

Hence,

$$\sum l(I_n) \leq mO < \infty .$$

Hence we can find an integer N such that

$$\sum_{N+1}^{\infty} l(I_n) < \frac{\varepsilon}{5} .$$

Let $R = E - \bigcup_{n=1}^{N} I_n$.

Claim: m*R < ε.

Let x be any arbitrary point of R.

Then $x \notin$ the closed set $\bigcup_{n=1}^{N} I_n$.

Hence we can find an interval I in J which contains x and whose length is so small that I does not meet any of the intervals $I_1, I_2, \ldots I_N$.

If I I_i =ϕ for i n,

Then we must have,

$l(I)$ $k_n < 2l(I_{n+1})$.

Since lim $l(I_n)$ =0,

the interval I must meet atleast one of the interals I_n.

Let n be the smallest integer such that I meets I_n.

Then n> N and $l(I)$ k_{n-1} $2l(I_n)$.

Since x is in I and I has a point in common with I_n,

it follows that the distance from x to the midpoint of I_n

is atmost $l(I)+\frac{1}{2}l(I_n)$ $2l(I_n) + \frac{1}{2}l(I_n)$

$$= \frac{5}{2}l(I_n) .$$

Let J_n be the interval having the same mid-point as I_n and five times its length.

Then $x \in J_n$.

Hence,

$$R \subset \bigcup_{N+1}^{\infty} J_n .$$

And m*R $\sum_{N+1}^{\infty} l(J_n)$)

$$= 5\sum_{N+1}^{\infty} l(I_n)$$

$< \varepsilon$.

Hence given ε >0, there is a finite disjoint collection { $I_1, I_2, \ldots I_N$} of intervals in J such that

$$m^*\left[E - \bigcup_{n=1}^{N} I_n \right] < \varepsilon.$$

Hence the theorem.

Definition

The derivatives of a function f at x is defined as follows:

$$D^{+} f(x) = \overline{\lim_{h \to 0+}} \frac{f(x+h) - f(x)}{h},$$

$$D^{-} f(x) = \overline{\lim_{h \to 0-}} \frac{f(x) - f(x-h)}{h}$$

$$D_{+} f(x) = \underline{\lim_{h \to 0+}} \frac{f(x+h) - f(x)}{h}$$

$$D_{-} f(x) = \underline{\lim_{h \to 0-}} \frac{f(x) - f(x+h)}{h}.$$

Remarks :

From the above definition, it is clear that

1. $D^{+} f(x) \quad D_{+} f(x)$
2. $D^{-} f(x) \quad D_{-} f(x)$.

Definition

If $D^{+} f(x) = D_{+} f(x) = D^{-} f(x) = D_{-} f(x) \neq \pm$ **,** then f is said to be differentiable at x and f'(x) is defined as the common value of the derivatives at x.

If $D^{+} f(x) = D_{+} f(x)$, then f is said to have a right hand derivative at x, denoted by f'(x+) and f'(x+) is defined as their common value.

If $D^{-} f(x) = D_{-} f(x)$, then f is said to have a left-hand derivative at x, denoted by f'(x-) and f'(x-) is defined as their common value.

Check your progress

Prove that if f is continuous on [a, b] and one of its derivatives (say D^{+}) is everywhere nonnegative on(a, b), then f is nondecreasing on [a, b].

i.e f(x) f(y) for x y.

Theorem 21.2

Let f be an increasing real-valued function on the interval [a, b].

Then f is differentiable almost everywhere.

The derivative f' is measurable, and

$$\int_a^b f'(x)dx \leq f(b) - f(a).$$

Proof

First let us show that the set of all points at which any of two derivatives are unequal has measure zero.

Let $E = \{ x \,/\, D^{+} f(x) > D^{-} f(x)\}$

Then E is the union of the sets $E_{u,v}$ where

$$E_{u,v} = \{ x \,/\, D^{+} f(x) > u > v > D^{-} f(x)\},$$

for all rationals u and v.

If we show that $m^* E_{u,v} = 0$,
Then $m^* \; E_{u,v} = 0$
and hence $m^*E=0$.
Similarly we can prove for all other cases.

Claim: $m^* E_{u,v} =0$
Let $s = m^* E_{u,v}$.
Choosing $\varepsilon > 0$, enclose $E_{u,v}$ in an open set O with $mO < s + \varepsilon$.
For each point x in $E_{u,v}$,
there is an arbitrarily small interval $[x - h, x]$ contained in O such that
$f(x) - f(x - h) < vh$. [since $D^- f(x) < v$]
By Vitali lemma,
We can choose a finite collection of intervals $\{ I_1, I_2, \ldots I_N\}$ of them whose interiors cover a subset A of $E_{u,v}$ of outer measure greater than $s - \varepsilon$.
Summing over these intervals, we get

$$\sum_{n=1}^{N}[f(x_n) - f(x_n - h_n)] < \sum_{n=1}^{N} vh_n$$

$$= v\sum_{n=1}^{N} h_n$$

$$< vmO$$

$$< v(s+\varepsilon). \qquad \ldots(1)$$

Again for each point y in A ,
there is an arbitralily small interval $(y, y+k)$ that is contained in some I_n and for which
$f(y+k) - f(y) > uk$, [since $D^+ f(x) > u$]
Hence , again by vitali Lemma,
there is finite collection $\{ J_1, J_2, \ldots J_M\}$ of such intervals such that there union contains a subset of A of outer measure greater that $s-2\varepsilon$.
Then summing over these intervals, we get

$$\sum_{i=1}^{M}[f(y_i + k_i) - f(y_i)] > \sum_{i=1}^{M} uk_i = u\sum_{i=1}^{M} k_i > u(s - 2\varepsilon). \qquad \ldots(2)$$

Since each interval J_i is contained in some I_n,
if we sum over those i for which $J_i \subset I_n$, we get
$[f(y_i + k_i) - f(y_i)]$ $[f(x_n) - f(x_n - h_n)]$ [since f is increasing]

Hence From (1) and (2),

$$v(s+\varepsilon) > \sum_{n=1}^{N}[f(x_n) - f(x_n - h_n)]$$

$$\sum_{i=1}^{M}[f(y_i + k_i) - f(y_i)]$$

$$>u(s-2\varepsilon)$$

i.e $v(s+\varepsilon) > u(s-2\varepsilon)$.
Since this is true for each $\varepsilon > 0$, this implies
$vs > us$.
If s 0, then we can cancel s on both sides and get
$v > u$,
which is a contradiction.

Hence s = 0.
i.e $m^* E_{u,v} = 0$.
Hence our claim is proved.
Hence

$$g(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

is defined almost everywhere and
f is differentiable wherever g is finite.

Let $g_n(x) = n[f(x+1/n) - f(x)]$,
where we set f(x) = f(b) for x b.

Then $g_n(x)› g(x)$ for almost all x.
Hence g is measurable.
Since f is increasing,

$$g_n \quad 0.$$

Hence by Fatou’s lemma,

$$\int_a^b g \le \underline{\lim}\int_a^b g_n = \underline{\lim}\, n\int_a^b [f(x+1/n) - f(x)]dx$$
$$= \underline{\lim}\left[n\int_a^b f(x+1/n)dx - n\int_a^b f(x)dx\right]$$
$$= \underline{\lim}\left[n\int_{a+1/n}^{b+1/n} f(x)dx - n\int_a^b f(x)dx\right]$$
$$= \underline{\lim}\left[n\int_b^{b+1/n} f - n\int_a^{a+1/n} f\right]$$
$$= \underline{\lim}\left[nf(b)[(b+1/n)-b] - n\int_a^{a+1/n} f\right]$$

[since f(x)=f(b) for x b]

$$f(b) - f(a).$$

Hence g is integrable.
Hence g is finite almost everywhere.
Hence f is differentiable a.e. and g=f ’ a.e.
Since g is measurable, f ’ is measurable and

$$\int_a^b f'(x)dx \le f(b) - f(a).$$

Hence the theorem.

Check your progress

1. Let f be a function defined by f(0) =0 and f(x) = x sin(1/x) for x 0 .
Find $D^+f(0)$, $D_+ f(0)$, $D^- f(0)$ and $D_-f(00$.
2. Show that $D^+[-f(x)] = -D_+f(x)$.
3. show that $D^+(f+g) \quad D^+f + D^+g$.

21.3 Let us sum up

In this lesson, you have studied

- The vitali covering of a set
- Vitali Lemma which states that a monotone function on an interval is differentiable and
- An increasing function on an interval is differentiable and its derivative is measurable

21.4 Lesson End Activities

1. Show that $D^+ (f+g) \leq D^+ (f) + D^+ (g)$

21.5 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

LESSON – 22

DIFFERENTIATION OF AN INTEGRAL AND ABSOLUTE CONTINUITY

Contents

22.0 Introduction

In this lesson we are going to study about functions of bounded variation , differentiation of an integral and absolute continuity of a function

22.1 Aims and Objectives

After studying this lesson, you would know

- The definition of function of bounded variation
- The definition of indefinite integral
- Differential of an integral
- Absolute continuity and
- Every absolutely continuous function is the indefinite integral of its derivative.

22.2 Functions of bounded variations

Definition: Function of bounded variation

Let f be a real valued function on the interval [a, b], and let $a = x_0 < x_1 < \ldots\ldots x_k = b$ be any subdivision of [a, b].

Define
$$p = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+$$

$$n = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^-$$

$$t = n + p = \sum_{i=1}^{k} | f(x_i) - f(x_{i-1}) |$$

Here we use the following notation :

$r^+ = r$, if r   0, otherwise r = 0.

Similarly $r^- = r$, if r   0, otherwise r = 0.

And $|r| = r^+ + r^-$.

Let P = sup p,
N = sup n,
T = sup t,

where we take the suprema over all possible subdivisions of [a, b].

Clearly P   T   P+N.

We call P, the positive variation of f over [a, b],
N, the negative variation of f over [a, b] and
T, the total variation of f over [a, b].

Since T depends on the interval [a, b] and the function f, we sometimes write T as T_a^b, or $T_a^b(f)$.

If T < , f is said to be of bounded variation over [a, b].

This can also be written as f $\in$ BV.

Remarks:

Here we say two important results on bounded variation without proof

1.A function f is of bounded variation on [a, b] if and only if f is the difference of two monotone functions an [a, b].

2.If f is of bounded variation on [a, b], then f'(x) exists for almost all x in [a, b].

22.3 Differentiation of an integral

Definition: Indefinite integral

If f is an integrable function on [a, b], the indefinite integral of f is defined to be the function F defined on [a, b] as

$$F(x) = \int_a^x f(t)dt$$

Theorem 22.1

If f is integrable on [a, b], then the function F defined by

$$F(x) = \int_a^x f(t)dt$$

is a continuous function of bounded variation on [a, b].

Proof

By theorem 19.7, F is a continuous function on [a, b]

To show that F is of bounded variation :

Let a = $x_0 < x_1 < \ldots\ldots x_k$ =b be any subdivision of [a, b].

Then T_a^b(F) = $\sum_{i=1}^{k} | F(x_i) - F(x_{i-1}) |$

$$= \sum_{i=1}^{k} | \int_{x_{i-1}}^{x_i} f(t)dt |$$

$$\sum_{i=1}^{k} \int_{x_{i-1}}^{x_i} | f(t) | dt$$

$$= \int_a^b | f(t) | dt \text{ .}$$

Hence ,

$$T_a^b(F) \quad \int_a^b | f(t) | dt < \text{ .}$$

Hence F is of bounded variation over [a, b].

Hence the theorem.

Theorem 22.2

If f is integrable on [a, b] and

$$\int_a^x f(t)dt = 0 \text{ for all } x \in [a, b],$$

then f(t) = 0 a.e in [a, b].

Proof

Suppose f(x) > 0 on a set E of positive measure.
Then by theorem 16.15,
there is a closed set $F \subset E$ with mF>0.

Let O = (a, b) –F .

Then either,

$$\int_a^b f \neq 0 \text{, or else}$$

$$0 = \int_a^b f = \int_F f + \int_O f .$$

Therefore,

$$\int_O f = - \int_F f \quad 0.$$

But since O is an open set , i
t is the disjoint union of a countable collection $\{(a_n, b_n)\}$ of open intervals.
Hence by theorem19.4,

$$\int_O f = \quad \int_{a_n}^{b_n} f \quad 0.$$

Hence for some n, we have

$$\int_{a_n}^{b_n} f \quad 0.$$

Therefore,

either $\int_a^{a_n} f \quad 0.$

Or $\int_a^{b_n} f \quad 0.$

Hence in any case,
if f is positive on a set of positive measure, then for some $x \in [a, b]$,

$$\int_a^x f(t)dt \quad 0.$$

Similarly we can prove for f negative on a set of positive measure.
Hence we are getting a contradiction.

Hence f(t) = 0 a.e on [a, b].

Theorem 22.3

If f is bounded and measurable on [a, b] and

$$F(x) = \int_a^x f(t)dt + F(a),$$

then F’(x) = f(x) for almost all x in [a, b].

Proof

By theorem 22.1,
F is of bounded variation over [a, b].
Hence F’(x) exists for almost all x in [a, b] and
F is continuous on [a, b].
Since f is bounded on [a, b],
$| f | \quad K$, for some real K.

Define $f_n(x) = \dfrac{F(x+h) - f(x)}{h}$ with h=1/n.

Hence ,

$$f_n(x) = \frac{1}{h}\int_x^{x+h} f(t)dt \qquad \text{with } h=1/n.$$

Hence,

$$| f_n(x) | =| \frac{1}{h}\int_x^{x+h} f(t)dt \,|$$

$$\frac{1}{h}\int_x^{x+h} | f(t) | \, dt$$

$$\frac{1}{h}\int_x^{x+h} K dt$$

$$= Kh/h$$

$$= K.$$

Hence ,

$$|f_n| \quad K.$$

Since $f_n(x)$› $F'(x)$ a.e.,

by the bounded convergence theorem, we get for $c \in [a, b]$,

$$\int_a^c F'(x)dx = \lim \int_a^c f_n(x)dx$$

$$= \lim_{h\to 0}\frac{1}{h}\int_a^c (F(x+h)-F(x))dx$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\int_a^c F(x+h)dx - \frac{1}{h}\int_a^c F(x)dx\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\int_{a+h}^{c+h} F(x)dx - \frac{1}{h}\int_a^c F(x)dx\right]$$

$$= \lim_{h\to 0}\left[\frac{1}{h}\int_c^{c+h} F(x)dx - \frac{1}{h}\int_a^{a+h} F(x)dx\right]$$

$$= F(c) - F(a)$$

$$= \int_a^c f(x)dx \ . \qquad \text{[Since F is continuous]}$$

Hence,

$$\int_a^c \{F'(x) - f(x)\}dx = 0 \text{ for all } c \in [a, b].$$

Hence by theorem 22.2

$$F'(x) - f(x) = 0 \text{ a.e on } [a, b].$$

Hence $\qquad F'(x) = f(x)$ a.e.

Hence the theorem.

Theorem 22.4

Let f be an integrable function on [a, b], and suppose that

$$F(x) = F(a) + \int_a^x f(t)dt \ .$$

Then $F'(x) = f(x)$ for almost all x in [a, b].

Proof

Without loss of generality assume that f 0.

Let f_n be defined as

$$f_n(x) = f(x) \qquad \text{if } f(x) \quad n,$$

$$\text{and} \quad f_n(x) = n \qquad \text{if } f(x) > n.$$

Hence,

$$f - f_n \quad 0.$$

Hence ,

$$G_n(x) = \int_a^x (f - f_n) \qquad \ldots(1)$$

is an increasing function of x.

Hence it has a derivative almost everywhere and this derivative is nonnegative.

[by theorem21.2]

Applying theorem 22.3 to the bounded and measurable functions f_n, we get

$$\frac{d}{dx}\int_a^x f_n = f_n(x) \text{ a.e.,}$$

From(1),

$$G_n(x) = \int_a^x f - \int_a^x f_n \ .$$

Therefore,

$$\int_a^x f = G_n(x) + \int_a^x f_n \ .$$

i.e $\quad F(x) - F(a) = G_n(x) + \int_a^x f_n \ .$

Hence,

$$F'(x) = \frac{d}{dx} G_n(x) + \frac{d}{dx}\int_a^x f_n$$

$$= \frac{d}{dx} G_n(x) + f_n(x) \text{ a.e.}$$

$$f_n(x) \text{ a.e.}$$

Since n is arbitrary,

$$F'(x) \quad f(x) \text{ a.e.}$$

Hence,

$$\int_a^b F'(x)dx \qquad \int_a^b f(x)dx = F(b) - F(a). \qquad \ldots(2)$$

But by theorem 21.2 ,

$$\int_a^b F'(x)dx \qquad F(b) - F(a). \qquad \ldots(3)$$

From (2) and (3), we get

$$\int_a^b F'(x)dx = F(b) - F(a) = \int_a^b f(x)dx$$

Hence ,

$$\int_a^b (F'(x) - f(x))dx = 0$$

Since $F'(x) - f(x) \quad 0$, this implies that

$$F'(x) - f(x) = 0 \text{ a.e.}$$

Hence,

$$F'(x) = f(x) \text{ a.e.}$$

Hence the theorem.

22.4 Absolute continuity

Definition

A real valued function f defined on [a, b] is said to be absolutely continuous on [a, b] if, given ε > 0, there is a δ > 0 such that

$$\sum_{i=1}^{n} | f(x_i') - f(x_i) | < \varepsilon$$

for every finite collection {(x_i, x_i')} of nonoverlapping intervals with

$$\sum_{i=1}^{n} | x_i' - x_i | < \delta.$$

Theorem 22.5

If f is absolutely continuous on [a, b], then it is of bounded variation on [a, b].

Proof

Since f is absolutely continuous on [a, b],
given ε =1, there exists a δ > 0 such that

$$\sum_{i=1}^{n} | f(x_i') - f(x_i) | < 1,$$

for every finite collection {(x_i, x_i')} of nonoverlapping intervals with

$$\sum_{i=1}^{n} | x_i' - x_i | < \delta.$$

Let K be the largest integer less than 1 + (b –a)/ δ.
Now any subdivision of [a, b] can be split into K sets of intervals,
each of total length less than δ, by inserting fresh division points, if necessary.
Hence for any subdivision,

$$t = \sum_{i=1}^{k} | f(x_i) - f(x_{i-1}) |$$

$$\sum_{i=1}^{K} | f(x_i) - f(x_{i-1}) |$$

<1

K.

i.e t K.

Hence ,

sup t K.

Hence,

T K.

Hence f is of bounded variation on [a, b].
Hence the theorem.

Check your progress

1. An absolutely continuous function is continuous.
2. The sum and difference of two absolutely continuous functions is absolutely continuous.
3. Prove that the product of two absolutely continuous functions is absolutely continuous.
4. Prove that if f is absolutely continuous, then f has a derivative almost everywhere.

Theorem 22.6

If f is absolutely continuous on [a, b] and f'(x) = 0 a.e., then f is constant.

Proof

To prove that f is constant ,
it is enough to show that f(a) = f(c) for any $c \in [a, b]$.

Let $c \in [a, b]$.
Let $E \subset (a, c)$ be the set of measure $c - a$ in which $f'(x) = 0$.
Let ε and η be arbitrary positive numbers.
Since f is absolutely continuous,
given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} | f(x_i') - f(x_i) | < \varepsilon \qquad \ldots(1)$$

for every finite collection $\{(x_i, x_i')\}$ of nonoverlapping intervals with

$$\sum_{i=1}^{n} | x_i' - x_i | < \delta.$$

To each x in E,
there is an arbitrarily small interval $[x, x+h]$ contained in $[a, c]$ such that

$$|f(x+h) - f(x)| < \eta h. \qquad \ldots(2)$$

Then by Vitali Lemma,
we can find a finite collection $\{ [x_k, y_k]\}$ of nonoverlapping intervals of this kind which cover all of E except for a set of measure less than δ.
If we label the x_k such that $x_k \quad x_{k+1}$, then

$$y_0 = a \quad x_1 < y_1 \quad x_2 < \ldots\ldots \quad y_n \quad c = x_{n+1}$$

and
$$\sum_{k=0}^{n} | x_{k+1} - y_k | < \delta.$$

From (2),

$$\sum_{k=1}^{n} | f(y_k) - f(x_k) | \quad \eta\ (y_k - x_k)$$
$$< \eta(c - a) \qquad \ldots(3)$$

From (1),

$$\sum_{k=0}^{n} | f(x_{k+1}) - f(y_k) | < \varepsilon \qquad \ldots(4)$$

Hence

$$|f(c) - f(a)| = \left| \sum_{k=0}^{n} [f(x_{k+1}) - f(y_k)] + \sum_{k=1}^{n} [f(y_k) - f(x_k)] \right|$$
$$\sum_{k=0}^{n} | f(x_{k+1}) - f(y_k) | + \sum_{k=1}^{n} | f(y_k) - f(x_k) |$$
$$\varepsilon + \eta(c - a).$$

Since ε and η are arbitrary positive numbers,
$f(c) - f(a) = 0$.
Hence $f(c) = f(a)$ for any $c \in [a, b]$.
Hence f is constant on $[a, b]$.
Hence the theorem.

Theorem 22.7

A function F is an indefinite integral if and only if it is absolutely continuous.

Proof

If F is an indefinite integral,
then by theorem 19.7, F is absolutely continuous.

Conversely,
assume that F is absolutely continuous on $[a, b]$.

Then F is of bounded variation on [a, b].
Hence F can be written as a difference of two monotone increasing functions, say F_1 and F_2.
i.e $F(x) = F_1(x) - F_2(x)$.
Since increasing functions are differentiable almost everywhere,
$F'(x) = F_1'(x) - F_2'(x)$ exists almost everywhere.
Hence ,

$$|F'(x)| \quad | F_1'(x)| + | F_2'(x) |.$$

Hence $\int | F'(x) | dx \quad \int | F_1'(x) | dx + \int | F_2'(x) | dx$

$$F_1(b) + F_2(b) - F_1(a) - F_2(a) \text{ [by theorem21.2]}$$

Hence $F'(x)$ is integrable.

Let $G(x) = \int_a^x F'(t)dt$.

Then G is absolutely continuous.
Hence,
f = F – G is also absolutely continuous.
By theorem 22.4,

$$G'(x) = F'(x) \text{ a.e.}$$

Hence,

$$f'(x) = 0 \text{ a.e.}$$

Hence by theorem 22.6,
f is constant and equal to F(a).
Hence,

$$F(x) = G(x) + F(a)$$

Hence,

$$F(x) = \int_a^x F'(t)dt + F(a).$$

Hence F is an indefinite integral.
Hence the theorem.

Theorem 22.8

Every absolutely continuous function is the indefinite integral of its derivative.

Proof

Let F be an absolutely continuous function on an interval [a, b]
Then by theorem 22.7,
F is an indefinite integral.
Hence there exists a integrable function f such that

$$F(x) = \int_a^x f(x)dx \qquad a \quad x \quad b.$$

Then by theorem 22.4,
$F'(x) = f(x)$.
Hence F is the indefinite integral of its derivative.
Hence the theorem.

22.5 Let us sum up

Thus in this lesson, we have seen

- The definition of function of bounded variation.
- The definition of indefinite integral.
- If a function f is integrable on [a, b], then its indefinite integral is a continuous function of bounded variation on [a, b].
- If f is bounded and measurable on [a, b], then the derivative of its indefinite integral is itself almost everywhere.

- The same is true if f is an integrable function on [a, b].
- The definition of Absolute continuity.
- If the derivative of an absolutely continuous function is zero almost everywhere, then it is constant.
- A function is an indefinite integral if and only if it is absolutely continuous and
- Every absolutely continuous function is the indefinite integral of its derivative.

22.6 Lesson End Activities

1. A monotone function f on [a,b] is called singular if f' = 0 a.e. Show that any monotonic increasing function is the sum of an Absolutely contionous function and a singular function.

22.7 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

LESSON – 23

THE MINKOWSKI AND HOLDER INEQUALITIES

Contents

23.0 Introduction

In this lesson, we are going to study about L^p spaces, The Minkowski and Holder Inequalities.

23.1 Aims and Objectives

After studying this lesson, you would know

- The definition of L^p spaces.
- The definition of norm in the L^p spaces.
- Convex functions
- The Minkowski Inequality and
- The Holder Inequality.

23.2 The L^p Spaces

Definition

Let p be a positive real number. A measurable function defined on [0, 1] is said to belong to the space $L^p = L^p[0, 1]$

$$\text{if } \int_0^1 |f|^p < \ .$$

Remarks:

1. L^1 consists precisely of the Lebesgue integrable functions on [0, 1].
2. Since $|f + g|^p \quad 2^p(|f|^p + |g|^p)$,
   Sum of two functions in L^p is again in L^p.
3. If $f \in L^p$,
   $$\text{then } \int_0^1 |f|^p < \ .$$
   Therefore for any scalar α,
   $$\int_0^1 |\alpha f|^p = |\alpha|^p \int_0^1 |f|^p < \ .$$
   Hence $\alpha f \in L^p$.
4. From 2 and 3, it follows that $\alpha f + \beta g \in L^p$,
   whenever $f \in L^p$ and $g \in L^p$, α and β, any scalars.
   Hence L^p spaces are linear spaces.

Definition

For a function $f \in L^p$, the norm of f is defined as

$$\| f \| = \| f \|_p = \left\{ \int_0^1 | f |^p \right\}^{1/p} .$$

Remarks

1. In the coming sections, we are going to prove that $\| f + g \| \quad \|f\| + \|g\|$.
2. A linear space is said to be a normed linear space, if for every f in the space, a number $\|f\|$ associated in such a way that
    (i) $\| f \| = 0$ if and only if $f \equiv 0$
    (ii) $\| \alpha f \| = | \alpha | \, \| f\|$
    (iii) $\| f + g \| \quad \| f \| + \| g \|$.
3. Norms on linear spaces do not satisfy property(i), because from $\|f\| = 0$, we can only conclude that $f = 0$ a.e.
   Hence we consider two measurable functions to be equivalent if they are equal almost everywhere.
   If we do not distinguish between equivalent functions, then L^p spaces are normed linear spaces.
4. L is used to denote the space of all bounded measurable functions on [0, 1].
5. The norm on L is defined as
   $\| f \| = \| f\| \quad = \text{ess sup} \, | f(t) |$,
   where ess sup f(t) is the infinimum of sup g(t), as the g ranges over all functions which are equal to f almost everywhere.
6. Hence ess sup $f(t) = \inf \{ M / \ m\{t / f(t) > M\} = 0 \}$.

Check your progress

1. Prove that $\|f\| = 0$ if and only if $f = 0$ a.e.
2. Prove that $\| \alpha f \| = | \alpha | \, \| f \|$.
3. Show that $\| f + g\| \quad \| f \| \ + \| g\|$.
4. Show that $\| f + g\|_1 \quad \| f \|_1 + \| g\|_1$.
5. Show that $\int fg \quad \| f \|_1 \| g\|$.

23.3 Convex functions

Definition

A function φ defined on an open interval (a, b) is said to be convex if for each x, y $\in$ (a, b) and each λ, 0 λ 1,

$\varphi(\lambda x + (1 - \lambda)y) \quad \lambda\varphi(x) + (1 - \lambda)\, \varphi(y)$.

Check your progress

Prove that $\varphi(t) = t^p$ is a convex function on [0,) for 1 p < .

23.4 The Minkowski Inequality

Theorem23.1 Minkowski Inequality for 1 p .

If f and g are in L^p with 1 p ,

then so is f+g and

$$\| f + g\|_p \quad \| f \|_p + \| g \|_p.$$

If 1 < p < ,

then equality can hold if there are nonnegative constants α and β such that

$$\beta f = \alpha g.$$

Proof

The result is trivial if $p =$, $\|f\| = 0$ or $\|g\| = 0$ as shown below:

Case (i) $p =$.

Since sup $(f + g)(t)$ sup $f(t)$ + sup $g(t)$,

ess sup $(f + g)(t)$ ess sup $f(t)$ + ess sup $g(t)$,

Hence $\|f + g\|$ $\|f\|$ + $\|g\|$.

Case (ii) $\|f\| = 0$ or $\|g\| = 0$

If $\|f\| = 0$,

then $f = 0$ a.e.,

Hence,

$f + g = g$ a.e

Hence,

$\|f + g\|_p = \|g\|_p$.

Similar result follows when $\|g\| = 0$.

Hence the inequality holds.

Now Assume that 1 $p <$ and $\|f\| = \alpha$ 0, $\|g\| = \beta$ 0.

Hence there are functions f_0 and g_0 such that

$|f| = \alpha f_0$ and $|g| = \beta g_0$,

and $\|f_0\| = \|g_0\| = 1$.

Let $\lambda = \alpha/(\alpha + \beta)$.

Then $(1 - \lambda) = 1 - \alpha/(\alpha + \beta)$

$= \beta/(\alpha + \beta)$.

Therefore,

$|f(x) + g(x)|^p$ $(|f(x)| + |g(x)|)^p$

$= [\alpha f_0(x) + \beta g_0(x)]^p$

$= [(\alpha + \beta)\lambda f_0(x) + (\alpha + \beta)(1 - \lambda) g_0(x)]^p$

$= (\alpha + \beta)^p[\lambda f_0(x) + (1 - \lambda) g_0(x)]^p$

$(\alpha + \beta)^p[\lambda f_0(x)^p + (1 - \lambda) g_0(x)^p]$ …(1)

[Since $\varphi(t) = t^p$ is a convex function on [0,) for 1 $p <$]

Integrating both sides of this inequality gives

$$\int_0^1 |f(x) + g(x)|^p\, dx \le (\alpha + \beta)^p \lambda \int_0^1 f_0(x)^p\, dx + (\alpha + \beta)^p (1 - \lambda) \int_0^1 g_0(x)^p\, dx$$

Hence ,

$\|f + g\|^p$ $(\alpha + \beta)^p[\lambda \|f_0\|^p + (1 - \lambda)\|g_0\|^p]$

$(\alpha + \beta)^p$ [since $\|f_0\| = \|g_0\| = 1$.]

$= (\|f\| + \|g\|)^p$

Taking p-th root on both sides, we get

$\|f + g\|$ $\|f\| + \|g\|$.

Hence the required result.

If $1 < p <$,

the inequality (1) is strict unless $f_0(x) = g_0(x)$ and sgn $f(x) =$ sgn $g(x)$.

i.e $f_0 = g_0$ a.e and sgn $f =$ sgn g a.e.

i.e $\beta f = \alpha g$.

Hence the theorem.

Remark:
The function $\varphi(t) = t^p$ is a concave function on [0,), for $0 < p < 1$.
Hence the proof above gives, mutalis mutandis, the following inequality

Minkowski Inequality for $0 < p < 1$.

Let f and g be two nonnegative functions which belong to the space L^p with $0 < p < 1$.
Then $\| f + g \| \quad \| f \| + \| g \|$.

23.5 The Holder Inequality

Theorem 23.2

Let $1 \quad p < \quad$. then for a, b, t nonnegative we have

$$(a + tb)^p \quad a^p + ptba^{p-1}.$$

Proof

Let $\varphi(t) = (a + tb)^p - a^p - ptba^{p-1}$.

Then $\varphi(0) = 0$,

And $\varphi'(t) = p(a+tb)^{p-1}b - pba^{p-1}$.

$= pb[(a+tb)^{p-1} - a^{p-1}] \quad 0$, for $p \quad 1$ and $a, b, t \quad 0$.

Hence,

$\varphi(t)$ is increasing and hence nonnegative for $t > 0$.

Hence,

$$(a + tb)^p - a^p - ptba^{p-1} \quad 0 \qquad \text{for a, b, t nonnegative.}$$

Hence,

$$(a + tb)^p \quad a^p + ptba^{p-1} \qquad \text{for a, b, t nonnegative.}$$

Hence the theorem.

Theorem 23.3 Holder Inequality:

If p and q are nonnegative extended real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

and if $f \in L^p$ and $g \in L^q$, then $f \cdot g \in L^1$ and

$$\int | fg | \quad \|f\|_p \|g\|_q.$$

Equality holds if and only if for some constants α and β, not both zero, we have $\alpha| f |^p = \beta| g |^q$ a.e.

Proof

First consider the case $p=1$ and $q=\;$.

Then $\int | fg | \quad (\sup |g|) \int | f |$

$\| g \| \quad \| f \|_1$.

Hence we get the required result.

Hence assume that $1< p < \quad$ and consequently $1 < q < \quad$.
It is sufficient to consider the case $f \quad 0$ and $g \quad 0$.
Otherwise if necessary, we can replace f by |f| and g by |g|.

Let $h(x) = g(x)^{q-1} = g(x)^{q/p}$. [since $q - 1 = q/p$.]

Hence ,

$$g(x) = h(x)^{p/q}$$
$$= h(x)^{p-1}.$$

Therefore,

$ptf(x)g(x) \quad = \quad ptf(x)h(x)^{p-1}$

$(h(x) + tf(x))^p - h(x)^p$ [by the above theorem]

Hence ,

$$pt \int fg \qquad \int |h+tf|^p - \int h^p$$

$$= \| h+ tf \|^p - \| h \|^p$$

$$(\| h \| + t\| f \|)^p - \| h \|^p \quad \text{[by Minkowski inequality]}$$

Differentiating both sides with respect to t , we get

$$p \int fg \qquad p(\| h \| + t\| f \|)^{p-1}\| f \|.$$

Hence at t=0, we get

$$p \int fg \qquad p\| h \|^{p-1}\| f \|.$$

Hence ,

$$\int fg \qquad \| h \|^{p-1}\| f \| = \| g \| \| f \|.$$

Hence we get the required result,

$$\int | fg | \qquad \|f \|_p \|g\|_q.$$

The Minkowski inequality used above is equal only if there exist nonnegative constants α and β such that

$$\beta|h| = \alpha|f| \text{ a.e.}$$

$$\text{i.e } \beta| g |^{q/p} = \alpha| f | \text{ a.e}$$

$$\text{i.e } \alpha| f|^p = \beta| g |^q.$$

Hence the theorem.

23.6 Let us sum up

In this lesson , we have studied

- L^p spaces .
- Norm on L^p Spaces
- Convex function
- Minkowski inequality and
- Holder inequality.

23.7 Lesson End Activities

if f : (a,b) $|\!\rightarrow$ R is a continous function such that $f((x+y)/2) \le (f(x) + f(y)) / 2 \ \forall \ x,y \in (a,b)$. Prove that f is convex.

23.8 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

LESSON - 24

CONVERGENCE AND COMPLETENESS

Contents

24.0 Introduction

In this lesson we are going to study about the convergence of a sequence of functions in a normed linear space, Cauchy sequence and Banach spaces.

24.1 Aims and Objectives

After studying this lesson , you would know

- Convergence of a sequence of functions in a normed linear space
- Pointwise convergence
- Cauchy sequence of functions
- Complete space
- Banach space
- Summable series and
- Riesz-Fischer Theorem.

24.2 Convergence and Completeness

Definition

A sequence $\{ f_n \}$ in a normed linear space is said to converge to an element f in the space if given $\varepsilon > 0$, there is an N such that for all $n > N$,

$$\| f - f_n \| < \varepsilon.$$

If f_n converges to f,

we write $f = \lim f_n$ or f_n › f .

Remark:

1. Another way of formulating the convergence of f_n to f is that:

f_n › f if $\| f_n - f \|$ › 0 .

2. Convergence in the space L^p is often referred to as: convergence in the mean of order p.

3. Hence a sequence of functions $\{ f_n \}$is said to converge to f in the mean of order p if each $f_n \in L^p$ and $\| f - f_n \|$ › 0 .

4. Convergence in L is nearly uniform convergence .

Definition

A sequence of functions $\{ f_n \}$ is said to converge pointwise to f if for each x,

$$f(x) = \lim f_n(x).$$

Definition

A sequence of functions $\{ f_n \}$ is said to converge to f almost everywhere if there is a set of measure zero such that for each x in $\tilde{E}$,

$$f(x) = \lim f_n(x).$$

Definition

A sequence $\{ f_n \}$ in a normed linear space is said to be a cauchy sequence if given $\varepsilon > 0$, there is an N such that for all n N and all m N,

$$\| f_n - f_m \| < \varepsilon.$$

Definition

A normed linear space is called complete if every Cauchy sequence in the space converges.

i.e, if for each Cauchy sequence $\{ f_n \}$ in the space there is an element f in the space such that f_n › f .

Definition

A complete normed linear space is called a Banach space.

Definition

A series $\{ f_n \}$ in a normed linear space is said to be summable to a sum s if s is in the space and the sequence of partial sums of the series converges to s.

i.e., $\| s - \sum_{i=1}^{n} f_i \|$ › 0 .

We write this as $s = \sum_{i=1}^{\infty} f_i$.

Definition

The series $\{ f_n \}$ is said to be absolutely summable if $\sum_{n=1}^{\infty} \| f_n \| < \infty$.

Remark:

1. For a series of real numbers, absolute summability implies that the series is summable. But this is in general not true fir series of elements in a normed linear space.
2. The following theorem shows that the implication holds when the space is complete.

Theorem 24.1

A normed linear space X is complete if and only if every absolutely summable series is summable.

Proof

Let X be a complete space.

To prove that every absolutely summable series is summable.

Let $\{ f_n \}$ be an absolutely summable series of elements of X.

i.e $\| f_n \| = M <$.

Hence for each $\varepsilon > 0$, there is an N such that

$$\sum_{n=N}^{\infty} \| f_n \| < \varepsilon.$$

Let $s_n = \sum_{i=1}^{n} f_i$ be the partial sum of the series $\{ f_n \}$.

Then for n m N,

$$\| s_n - s_m \| = \left\| \sum_{i=m}^{n} f_i \right\|$$

$$\sum_{i=m}^{\infty} \| f_i \|$$

$$\sum_{n=N}^{\infty} \| f_n \|$$

$$< \varepsilon.$$

Hence the sequence $\{ s_n \}$ of partial sums is a Cauchy sequence in X.

Since X is complete,

$\{ s_n \}$converges to an element s in X.

Hence $\{ f_n \}$ is summable.

Conversely,

Assume that every absolutely summable series is summable.

To prove that X is complete.

Let $\{ f_n \}$ be a Cauchy sequence in X.

Then by definition,

for each integer k, there is an integer n_k such that

$\| f_n - f_m \| < 2^{-k}$, for all n n_k and m n_k..

The n_k's may be chosen such that $n_{k+1} > n_k$.

Hence $\{ f_{n_k} \}_{k=1}^{\infty}$ is a subsequence of $\{ f_n \}$.

Let $g_1 = f_{n_1}$, and

$g_k = f_{n_k} - f_{n_{k-1}}$ for $k > 1$.

Then,

$$\sum_{i=1}^{k} g_i = \sum_{i=1}^{k} (f_{n_i} - f_{n_{i-1}}) = f_{n_k} .$$

Hence,

$\{ g_k \}$ is a series whose k-th partial sum is f_{n_k} .

But ,

$$\| g_k \| = \| f_{n_k} - f_{n_{k-1}} \|$$

$$2^{-k+1} \qquad \text{if } k > 1.$$

Hence,

$$\| g_k \| \quad \| g_1 \| + \quad 2^{-k+1}$$

$$= \| g_1 \| + 1.$$

Hence the series $\{ g_k \}$ is absolutely summable.

Hence by assumption,

$\{ g_k \}$ is also summable.

Hence there is an element f in X to which the partial sums of the series converge.

Hence the subsequence $\{ f_{n_k} \}$ converges to f.

Claim: $f = \lim f_n$.

Since $\{ f_n \}$ is a Cauchy sequence ,

for given $\varepsilon > 0$, there is an N such that for all n N and m N,

$\| f_n - f_m \| < \varepsilon/2$.

Since f_{n_k} › f ,

for the same ε > 0, there is a K such that for all k K,

$$\| f_{n_k} - f \| < \varepsilon/2.$$

Choose k large such that k > K and n_k N.

Then for n > N,

$$\| f_n - f \| = \| f_n - f_{n_k} + f_{n_k} - f \|$$

$$\| f_n - f_{n_k} \| + \| f_{n_k} - f \|$$

$$< \varepsilon/2 + \varepsilon/2$$

$$= \varepsilon.$$

Hence,

f_n › f .

Hence every Cauchy sequence of points of X converges to an element in X.

Hence X is complete.

Hence the theorem.

Theorem 24.2 Riesz-Fischer Theorem:

The L^p spaces are complete.

Proof

Assume that 1 p < .

By theorem 24.1, we need only to show that every absolutely summable series is summable in L^p.

Let { f_n } be an absolutely summable series in L^p.

Let $\sum_{n=1}^{\infty} \| f_n \| = M <$.

Define the functions g_n as

$$g_n(x) = \sum_{k=1}^{n} | f_k(x) | .$$

By Minkowski inequality,

$$\| g_n \| \quad \sum_{k=1}^{n} \| f_k \|$$

$$\sum_{n=1}^{\infty} \| f_n \|$$

$$= M.$$

i.e $\| g_n \|$ M.

Hence,

$$\int (g_n)^p \quad M^p .$$

For each x,

{ $g_n(x)$ } is an increasing sequence of extended real numbers

And hence,

{ $g_n(x)$ } must converge to an extended real number.

Let it be g(x).

The function g so defined is measurable and

since g_n 0, by Fatou's Lemma

$$\int g^p \quad \underline{\lim} \int (g_n)^p$$

$$M^p .$$

Hence,

g^p is integrable.

Hence g(x) is finite for almost all x.

For each x such that g(x) is finite,

the series $\sum_{k=1}^{\infty} f_k(x)$ is an absolutely summable series of real numbers.

Hence it must be summable to a real number.
Define this number as s(x).
Define s(x) = 0 for those x where g(x) = .
Hence,
the function s defined is the limit of the partial sums

$s_n = \sum_{k=1}^{n} f_k$ almost everywhere.

Hence,

s is measurable.

Since $|s_n(x)| \quad g(x)$,
$|s(x)| \quad g(x)$.

Hence s is in L^p and

$$|s_n(x) - s(x)|^p \quad (|s_n(x)| + |s(x)|)^p$$
$$(2g(x))^p$$
$$= 2^p[g(x)]^p.$$

Since $2^p g^p$ is integrable and $|s_n(x) - s(x)|^p$ converges to 0 for almost all x,

$$\int |s_n - s|^p \rightarrow 0 \text{ [by Lebesgue convergence theorem]}$$

Hence ,

$\|s_n - s\|^p$ › 0 .

Hence,

$\|s_n - s\|$ › 0 .

Hence the series $\{ f_n \}$ has the sum s in L^p.
Hence every absolutely summable series is summable in L^p.
Hence the L^p spaces are complete.
Hence the theorem.

Check your progress

1. Prove that each convergent sequence is a Cauchy sequence.
2. Prove that L   is complete.

24.3 Let us sum up

In this lesson, we have seen

- The definition of Convergence of a sequence of functions in a normed linear space
- The definition of Pointwise convergence
- The definition of Cauchy sequence of functions
- The definition of Complete space
- The definition of Banach space
- The definition of Summable series
- A space is complete if and only if every absolutely summable series is summable and
- The L^p spaces are complete.

24.4 Lesson End Activities

1. Prove that the space l_p is a Banach spalce given l_p ($l \le p < \propto$) consist of all real sequences (a_n)

$$\sum_{n=i}^{\propto_i} |a_n|^p < \propto \; \& \; ||\{a_n\}||_p = (\sum_{n=i}^{\propto_i} |a_n|^p)^{yp}$$

24.5 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.

LESSON - 25

BOUNDED LINEAR FUNCTIONALS ON THE L^p SPACES.

25.0 Introduction

In this lesson, we are going to study about bounded linear functionals on the L^p spaces.

25.1 Aims and Objectives

After studying this lesson, you would know

- Linear functionals
- Bounded linear functionals
- Bounded linear functionals on L^p spaces and
- Riesz Representation theorem.

25.2 Bounded linear functionals on the L^p spaces.

Definition

A linear functional on a normed linear space X is defined to be a mapping F of the space X into the set of real numbers such that

$$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g)$$

where f, g $\in$ X and α, β are scalars.

Definition

A linear functional f is said to be bounded, if there is a constant M such that

$$|F(f)| \quad M\|f\|, \qquad \text{for all } f \text{ in } X.$$

Definition

The smallest constant M for which the inequality

$$|F(f)| \quad M\|f\|, \qquad \text{for all } f \text{ in } X,$$

is true is called the norm of F.

Hence,

$$\|F\| = \sup \left\{ \frac{|F(f)|}{\|f\|} \right\},$$

as f ranges over all nonzero elements of X.

Theorem 25.1

Each function g in L^q defines a bounded linear functional F on L^p by

$$F(f) = \int fg .$$

And $\| F \| = \| g \|_q$.

Proof

Let g be a function in L^q.

Define a functional F on L^p by

$$F(f) = \int fg , \quad \text{for all f in } L^p.$$

Claim: F is a bounded linear functional on L^p.

For $f_1 , f_2 \in L^p$, and for scalars α, β ,

consider ,

$$\begin{aligned} F(\alpha f_1 + \beta f_2) &= \int(\alpha f_1 + \beta f_2)g \\ &= \alpha\int f_1 g \ + \beta \int f_2 g \quad \text{[by the properties of integrals]} \\ &= \alpha F(f_1) + \beta F(f_2). \end{aligned}$$

Hence F is linear on L^p.

$$\text{Consider} \quad |F(f) | \ = |\int fg | \qquad \int| fg | \qquad \| f \|_p \| g \|_q \qquad \text{[by Holder inequality]}$$

Hence,

$$\frac{|F(f)|}{\| f \|} \quad \| g \|_q.$$

Since this is true for all f in L^p,

$$\| F\| = \sup \left\{ \frac{|F(f)|}{\| f \|} \right\} \qquad \| g \|_q.$$

Hence,

F is a bounded linear functional with

$$\| F\| \quad \| g \|_q. \qquad \dots(1)$$

To prove the opposite inequality,

Let $f = | g |^{q/p} \text{sgn } g$, for $1< p <$,

where sgn is a function defined as

sgn x = 1 if x > 0,
sgn 0 = 0, and
sgn x= -1 if x < 0.

Therefore ,

$$\begin{aligned} | f |^p &=| | g |^{q/p}\text{sgn } g |^p \\ &= | g |^q \end{aligned}$$

And

$$\begin{aligned} fg &= (| g |^{q/p}\text{sgn } g)g \\ &=| g |^{q/p+1} \\ &=|g|^q. \quad \text{[since 1/p +1/q =1]} \end{aligned}$$

Hence $| f |^p =| g |^q =fg$ …(2)

Hence $\int| f |^p = \int fg$.

Hence f is in L^p.

And $\| f\|_p = (\int | f |^p)^{1/p}$

$= (\int | g |^q)^{1/p}$ [from (2)]

$= (\| g \|_q)_q^{/p}$.

Therefore ,

$$F(f) = \int fg$$

$$= \int | g |^q \qquad \text{[from (2)]}$$

$$= (\| g \|_q)^q$$
$$= \| g \|_q (\|g\|_q)^{q-1}$$
$$=\| g \|_q (\|g\|_q)^{q/p}$$
$$= \| g \|_q \|f\|_p.$$

Hence ,

$$\frac{| F(f) |}{\| f \|_p} = \| g \|_q.$$

Hence,

$\|F\|$ must be atleast as great as $\|g\|_q$. …(3).

From (1) and (3), we get

$$\| F \| = \| g \|_q.$$

Hence the theorem.

Theorem 25.2

Let g be an integrable function on [0, 1], and suppose that there is a constant M such that

$$| \int fg | \quad M\| f \|_p$$

for all bounced measurable functions f.

Then g is in L^q, and $\| g \|_q \quad M$.

Proof

Assume that $1 < p <$.

Define a sequence of bounded measurable functions g_n as

$$g_n(x) = \begin{cases} g(x) & if \quad | g(x) | \le n \\ 0 & if \quad | g(x) | > n \end{cases}$$

Then $\{ g_n \}$ is a sequence of bounded measurable functions.

Let $f_n = | g_n |^{q/p} \operatorname{sgn} g_n$.

Then $| f_n |^p = | | g_n |^{q/p} \operatorname{sgn} g_n|^p$

$= | g_n |^q$.

Therefore,

$$(\|f_n \|_p)^p = \int | f_n |^p$$

$$= \int | g_n |^q$$

$$=(\| g_n \|_q)^q$$

Hence,

$$\| f_n \|_p = (\| g_n \|_q)^{q/p} \qquad ...(1)$$

Also,

$$f_n \, g_n = (| g_n |^{q/p} \operatorname{sgn} g_n) \, g_n$$

$$= |g_n|^{q/p}\,|g_n|$$
$$= |g_n|^{q/p+1}$$
$$= |g_n|^{q}.$$

Hence $|g_n|^q = f_n g_n = f_n g.$ …(2)

Therefore,

$$\int |g_n|^q = \int f_n g$$

$$M\,\|f_n\|_p \qquad \text{[by assumption]}$$

$$= M(\|g_n\|_q)^{q/p} \qquad \text{[From (1)]}$$

i.e $(\|g_n\|_q)^q \quad M(\|g_n\|_q)^{q/p}.$

Hence,

$$(\|g_n\|_q)^{q-q/p} \quad M.$$

Hence,

$$\|g_n\|_q \quad M \qquad \text{[since } 1/p+1/q=1,\ \ q-q/p=1]$$

Therefore,

$$\int |g_n|^q \quad M^q.$$

Since $|g_n|^q$ conveges to $|g|^q$ a.e.,

By Fatou's Lemma,

$$\int |g|^q \quad \underline{\lim} \int |g_n|^q$$
$$M^q.$$

Hence ,

$$g \in L^q \text{ and } \|g\|_q \quad M.$$

Hence the theorem is proved for $1 < p <$.

If p=1,

Let $E = \{ x / |g(x)| \quad M+\varepsilon \}$

Let $f = (\mathrm{sgn}\, g)\chi_E.$

Then $|f| = \chi_E.$

Therefore,

$$\|f\|_1 = \int |f| = \int \chi_E = mE.$$

Hence $MmE = M\|f\|_1$

$$\int fg \qquad \text{[by assumption]}$$

$$(M+\varepsilon)mE.$$

Since ε was arbitrary,

this implies $mE = 0$.

Hence ,

$$g \in L \quad \text{and} \quad \|g\| \quad M$$

Hence we have proved the theorem for p =1.

Hence the theorem.

Theorem 25.3 Riesz Representation theorem.

Let F be a bounded linear functional on L^p, $1 \quad p <$.

Then there is a function g in L^q such that

$$F(f) = \int fg\,.$$

Also $\|F\| = \|g\|_q$.

Proof

Let χ_s be the characteristic function of the interval [0, s].

Since F is a bounded linear functional,

for each s, $F(\chi_s)$ is a real number.

Define $\Phi(s)$ as this number.

Hence Φ defines a function on [0, 1].

Claim: Φ is absolutely continuous.

Let $\{ (s_i, s_i') \}$ be any finite collection of nonoverlapping subintervals of [0, 1] of total length less than δ.

Then $\sum_i |\Phi(s_i') - \Phi(s_i)|$ =F(f), where

f = $\sum_i (\chi_{s_i'} - \chi_{s_i}) \operatorname{sgn}(\Phi(s_i') - \Phi(s_i))$.

Hence $(\| f \|_p)^p = \int | f |^p$

$$\sum_i (s_i' - s_i)$$

$< \delta$.

Hence $\sum_i |\Phi(s_i') - \Phi(s_i)|$ = F(f)

$\| F \| \, \| f \|_p$

$< \| F \| \, \delta^{1/p}$.

Therefore,

if we choose $\delta = \varepsilon^p / \|F\|^p$, then

$$\sum_i |\Phi(s_i') - \Phi(s_i)| < \varepsilon.$$

Hence Φ is absolutely continuous.

Hence by theorem 22.7,

Φ is an indefinite integral.

i.e there is an integrable function g on [0, 1] such that

$$\Phi(s) = \int_0^s g$$

Hence

$$\begin{aligned} F(\chi_s) &= \Phi(s) \\ &= \int_0^s g \\ &= \int_0^1 g \cdot \chi_s . \end{aligned}$$

Since every step functionon [0,1] is equal almost everywhere to a suitable linear combination $\sum c_i \chi_{s_i}$,

by the linearity of F and the integral, we get

$$F(\psi) = \int_0^1 g\psi$$

for each sep function ψ.

Let f be a bounded measurable function on [0, 1].

By theorem17.6,

there is a bounded sequence $\{ \psi_n \}$ of step functions which converge almost everywhere to f.

Hence the sequence $\{ | f - \psi_n|^p \}$ is uniformly bounded and

$$| f - \psi_n|^p \ › \ 0 \text{ a.e.}$$

Hence by bounded convergence theorem,

$$\| f - \psi_n \|^p › \ 0 .$$

Since F is bounded, and

$$| F(f) - F(\psi_n) | = | F(f - \psi_n) |$$

$$\quad \| F \| \, \| f - \psi_n \|_p .$$

Hence,

$$| F(f) - F(\psi_n) | \ › \ 0 .$$

Hence,

$$F(f) \ = \lim F(\psi_n).$$

Since $g\psi_n$ is always less than $| g |$ times the uniform bound for the sequence $\{ \psi_n \}$, by bounded convergence theorem,

$$\int fg \ = \lim \int g\psi_n .$$

Hence ,

$$\int fg \ = \lim F(\psi_n)$$

$$= F(f),$$

for each bounded measurable function f.

Since $| F(f) | \quad \| F \| \, \| f\|_p,$

$$| \int fg | \quad \| F \| \, \| f\|_p.$$

Hence by theorem 25.2,

$$g \in L^q \text{ and } \| g \|_q \quad \| F \|.$$

Claim: $F(f) = \int fg$, for each f in L^p.

Let f be an arbitrary function in L^p.

Then for each $\varepsilon > 0$, there exists a step function ψ such that

$$\| f - \psi \|_p < \varepsilon.$$

Since ψ is bounded,

$$F(\psi) = \int \psi g .$$

Hence

$$| F(f) - \int fg | = | F(f) - F(\psi) + \int \psi g - \int fg |$$

$$| F(f - \psi) | + | \int (\psi - f)g |$$

$$\| F \| \, \| f - \psi\|_p \ + \| g \|_q \| f - \psi\|_p$$

$$< (\| F \| + \| g \|_q) \| f - \psi\|_p$$

$$(\| F \| + \| g \|_q) \varepsilon.$$

Since ε is an arbitrary number,

$$F(f) = \int fg .$$

Hence by theorem 25.1,

$$\| F \| = \| g \|_q.$$

Hence, we have proved that there is a function g in L^q such that

$$F(f) = \int fg .$$

and $\| F \| = \| g\|_q$.

Hence the theorem.

25.3 Let us sum up

Thus in this lesson, we have seen,

- The definition of a linear functional on a normed linear space.
- Bounded linear functionals on L^p
- For Each function g in L^q , there is a bounded linear functional F on L^p by

  $F(f) = \int fg$ such that $\| F \| = \| g \|_q$.
- The converse of this is also true and it is given by Riesz Representation theorem.

25.4 Lesson End Acitivities

1. Give the representation for the bounded linear functionals on l_p $(1 \leq p < \propto)$

25.6 References

1. R.G. Bartle, Elements of Real Analysis, 2nd Edition, John Willy and Sons, New York, 1976.
2. W.Rudin, Real and Complex Analysis, 3rd Edition, McGraw-Hill, New York.