TEXT BOOK OF TRIGONOMETRY, VECTOR CALCULUS AND ANALYTICAL GEOMETRY

Contents

Unit I

Lesson - 1

Trigonometry

Contents

1.0 Aims and Objectives

We shall study the expansions of cosnx and sinnx by using the concept of Demoivre's Theorem, the concept of combinations and the concept of the Binomial expansion.

1.1 Expansions

1.1.1. Expansions of cos nθ and sin nθ

We know that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$\cos n\theta + i\sin n\theta = (\cos\theta + i\sin\theta)^n$$

$$= \cos^n\theta + nc_1\cos^{n-1}\theta\ \sin\theta + nc_2\cos^{n-2}\theta\,(i\sin\theta)^2 +$$

$$nc_3\cos^{n-3}\theta\,(i\sin\theta)^3 + nc_4\cos^{n-4}\theta\,(i\sin\theta)^4 + \ldots\ldots\ldots\ldots$$

$$= \cos^n\theta + i\,nc_1\cos^{n-1}\theta\ \sin\theta - nc_2\cos^{n-2}\theta\ \sin^2\theta - i\,nc_3\cos^{n-3}\theta$$

$$\sin^3\theta + nc_4\cos^{n-4}\theta\ \sin^4\theta + i\,nc_5\cos^{n-5}\theta\ \sin^5\theta\ \ldots\ldots\ldots..$$

$$= \cos^n\theta - nc_2\cos^{n-2}\theta\ \sin^2\theta + nc_4\cos^{n-4}\theta\ \sin^4\theta\ \ldots\ldots\ldots\ldots\ldots$$

$$+ i\,(nc_1\cos^{n-1}\theta\ \sin\theta - nc_3\cos^{n-3}\theta\ \sin^3\theta + nc_5\cos^{n-5}\theta$$

$$\sin^5\theta\ \ldots.)$$

Equate real and imaginary parts

$\text{Cos } n\theta = \cos^n\theta - nc_2 \cos^{n-2}\theta \sin^2\theta + nc_4 \cos^{n-4}\theta \sin^4\theta \ldots\ldots\ldots\ldots\ldots$

$\text{Sin } n\theta = nc_1 \cos^{n-1}\theta \sin\theta - nc_3 \cos^{n-3}\theta \sin^3\theta + nc_5 \cos^{n-5}\theta \sin^5\theta \ldots\ldots\ldots$

1.2 Examples

(1) Expand cos 6θ inpowers of cosθ

Or

Prove that

$\text{Cos } 6\theta = 32 \cos^6\theta + 48 \cos^4\theta + 18 \cos^2\theta - 1$

Proof: we know that

$\text{Cos } {}^n\theta = \cos n\theta - nc_2 \cos^{n-2}\theta \sin^2\theta + nc_4 \cos^{n-4}\theta \sin^4\theta \ldots\ldots$

Put n = 6,

$\text{Cos } 6\theta = \cos^6\theta - 6c_2 \cos^4\theta \sin^2\theta + 6c_4 \cos^2\theta \sin^4\theta - 6c_6 \sin^6\theta$

$$= \cos^6\theta - \frac{6.5}{1.2}\cos^4\theta \sin^2\theta + \frac{6.5.4.3}{1.2.3.4}\cos^2\theta \sin^4\theta - \sin^6\theta$$

$= \cos^6\theta - 15 \cos^4\theta \sin^2\theta + 15\cos^2\theta \sin^4\theta - \sin^6\theta$

$= \cos^6\theta - 15 \cos^4\theta (1 - \cos^2\theta) + 15 \cos^2\theta (1 - \cos^2\theta)^2 - (1 - \cos 2\theta)^3$

$= \cos^6\theta - 15 \cos^4\theta + 15 \cos^6\theta + 15 \cos^2\theta (1 + \cos^4\theta - 2 \cos^2\theta) -$

$(1 - \cos^6\theta - 3 \cos^2\theta + 3 \cos^4\theta)$

$= \cos^6\theta - 15 \cos^4\theta + 15 \cos^6\theta + 15 \cos^2\theta + 15 \cos^6\theta - 30 \cos^4\theta -$

$1 + \cos^6\theta + 3 \cos^2\theta - 3 \cos^4\theta$

$= 32 \cos^6\theta - 48 \cos^4\theta + 18 \cos^2\theta - 1$

2. Expand $\frac{\sin 6\theta}{\sin\theta}$ in powers of cosθ

Prove that $\frac{\sin 6\theta}{\sin\theta} = 32 \cos^5\theta - 32 \cos^3\theta + 6 \cos\theta$

Proof : We know that

$\text{Sin } n\theta = nc_1 \cos^{n-1}\theta \sin\theta - nc_3 \cos^{n-3}\theta \sin^3\theta + nc_5 \cos^{n-5}\theta \sin^5\theta \ldots\ldots\ldots$

Put n = 6

$$\text{Sin } 6\theta = 6c_1 \cos^5\theta \sin\theta - 6c_3 \cos^3\theta \sin^3\theta + 6c_5 \cos^5\theta \sin^5\theta$$

$$= 6 \cos^5\theta \sin\theta - \frac{6.5.4.}{1.2.3.} \cos^3\theta \sin^3\theta + 6 \cos\theta \sin^5\theta$$

$$= 6 \cos^5\theta \sin\theta - 20 \cos^3\theta \sin^3\theta + 6 \cos\theta \sin^5\theta$$

$$\frac{\sin 6\theta}{\sin\theta} = 6 \cos^5\theta - 20 \cos^3\theta \sin^2\theta + 6 \cos\theta \sin^4\theta$$

$$= 6 \cos^5\theta - 20 \cos^3\theta + 20 \cos^5\theta + 6 \cos\theta (1+\cos^4\theta - 2\cos^2\theta)$$

$$= 6 \cos^5\theta - 20 \cos^3\theta + 20 \cos^5\theta + 6 \cos\theta + 6 \cos^5\theta - 12 \cos^3\theta$$

$$= 32 \cos^5\theta - 32 \cos^3\theta \; 6 \cos\theta$$

1.3 Let us sum up

So far we have seen the expansion of $\cos n\theta$ and $\sin n\theta$ using Binomial theorem, Demoivre's theorem and concept of $i^2=-1$

1.4 Check your progress

1. find sin 2x and cos2x

1.5 Lesson End activities

Prove that $\frac{\sin 7\theta}{\sin\theta} = 64 \cos^6\theta - 80 \cos^4\theta + 24 \cos^4\theta - 1$

1. Prove that $\cos 4\theta = 8 \cos^4\theta - 8 \cos^2\theta + 1$
2. Prove that $\cos 7\theta = \cos 7\theta - 21 \cos^5\theta \sin^2\theta + 35 \cos^3\theta \sin^2\theta - 7\cos\theta \sin^6\theta$
3. Prove that $\frac{\sin 6\theta}{\sin\theta} = 32 \cos^5\theta - 32 \cos^3\theta + 6 \cos\theta$

1.6 Point for discussion

1. Prove that

$$\cos 7\theta \sec\theta = 64 \cos^6\theta + 114 \cos^4\theta + 56 \cos^2\theta - 7$$

1.7 References

1. Trigonometry by S. Narayanan

Lesson – 2

HYPERBOLIC FUNCTIONS

CONTENTS

2.0 Aims and Objectives

Our aim is to define the hyperbolic cosines of x and series of x using exponentiation.

Definition 1 : The hyperbolic cosine of x is defined as coshx = $\frac{e^{x}+e^{-x}}{2}$

Definition 2: The hyperbolic sine of x is defined as sinhx = $\frac{e^{x}-e^{-x}}{2}$

Definition 3: tanhx = $\frac{e^{x}+e^{-x}}{e^{x}+e^{-x}}$

More results

1. sin (i x) = i sin hx
2. cos (i x) = cos hx
3. tan (i x) = i tan hx

2.2. Examples

Separate into real and imaginary parts

a) sin (α+iβ)

sin (α+iβ) = sin α cos (iβ) + cos α sin (iβ)

= sin α cos hβ + cos α i sin hβ

= sin α cos hβ + i cos α sin hβ

Real part = sin α cos hβ

Imaginary part = cos α sin hβ

b) sin (α-iβ) = sin α cos (iβ) - cos α sin (iβ)

= sin α cos hβ - cos α i sin hβ

= sin α cos hβ - i cos α sin hβ

Real part = sin α cos hβ

Imaginary part = - cos α sin hβ

c) cos (α+iβ)

cos (α+iβ) = cos α cos (iβ) - sin α sin (iβ)

= cos α cos hβ - sin α (i sin hβ)

= cos α cos hβ - i sin α sin hβ

Real part = cos α cos hβ

Imaginary part = - sin α sin hβ

d) cos (α-iβ)

cos (α-iβ) = cos α cos (iβ) + sin α sin (iβ)

= cos α cos hβ + sin α (i sin hβ)

= cos α cos hβ + i sin α sin hβ

Real part = cos α cos hβ

Imaginary part = sin α sin hβ

e) tan (α+iβ) = $\dfrac{\sin(\alpha + i\beta)}{\cos(\alpha + i\beta)}$

multiply the numerator and denominator by 2 cos (α-iβ)

tan (α+iβ) = $\dfrac{2\sin(\alpha + i\beta)\cos(\alpha - i\beta)}{2\cos(\alpha + i\beta)\cos(\alpha - i\beta)}$ ---------(1)

numerator = 2 sin (α+iβ) cos (α-iβ)

= 2 sin A cos B; A = α+iβ

B = α-iβ

= sin (A+B) + sin (A-B)

= sin 2 α + sin (2iβ)

$= \sin 2\alpha + i \sinh 2\beta$

Denominator $= 2\cos(\alpha+i\beta)\cos(\alpha-i\beta)$

$= 2\cos A \cos B;\ A = \alpha+i\beta$

$B = \alpha-i\beta$

$= \cos(A+B) + \cos(A-B)$

$= \cos 2\alpha + \cos(2i\beta)$

$= \cos 2\alpha + \cos h2\beta$

Using in 1

$$\tan(\alpha+i\beta) = \frac{\sin 2\alpha + i\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}$$

$$= \frac{\sin 2\alpha}{\cos 2\alpha + \cosh 2\beta} + i\frac{\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}$$

$$\text{Real part} = \frac{\sin 2\alpha}{\cos 2\alpha + \cosh 2\beta}$$

$$\text{Imaginary part} = \frac{\sinh 2\beta}{\cos 2\alpha + \cosh 2\beta}$$

f) sin h(α+iβ)

$$\sin h(\alpha+i\beta) = \frac{1}{i}\left[i\sinh(\alpha + i\beta)\right]$$

$$= \frac{1}{i}\left[\sin i(\alpha + i\beta)\right]$$

$$= \frac{1}{i}\left[\sin(i\alpha - \beta)\right]$$

$$= -i\left[\sin(i\alpha)\cos\beta - \sin\beta\cos(i\alpha)\right]$$

$$= -i\left[i\sinh\alpha\cos\beta - \cosh\alpha\sin\beta\right]$$

$$= \sinh\alpha\cos\beta + i\cosh\alpha\sin\beta$$

Real part = $\sin h\alpha \cos\beta$;

Imaginary part = $\cos h\alpha \sin\beta$

g) cosh (α+iβ) = cos [i(α+iβ)]

= cos (iα-β)

$= \cos (i\alpha) \cos \beta + \sin (i\alpha) \sin \beta$

$= \cosh \alpha \cos \beta + i \sinh \alpha \sin \beta$

$= \cos \alpha \cos h\beta - i \sin \alpha \sin h\beta$

Real part $= \cosh \alpha \cos \beta$

Imaginary part $= \sinh \alpha \sin \beta$

h) $\tanh (\alpha+i\beta) = \frac{1}{i}[i \tanh(\alpha + i\beta)]$

$= \frac{1}{i}[\tan i(\alpha + i\beta)]$

$= \frac{1}{i}[\tan(i\alpha - \beta)]$

$= -i \tan (i\alpha-\beta)$

$= -i \frac{\sin(i\alpha - \beta)}{\cos(i\alpha - \beta)}$

$= i \frac{2\cos(\beta + i\alpha)\sin(\beta - i\alpha)}{2\cos(\beta + i\alpha)\cos(\beta - i\alpha)}$ ------- (1)

Numerator $= 2\cos(\beta + i\alpha)\sin(\beta - i\alpha)$

$= 2 \cos A \sin B;\ A = \beta+i\alpha$

$B = \beta-i\alpha$

$= \sin (A+B) - \sin (A-B)$

$= \sin 2 \beta - \sin (2i \alpha)$

$= \sin 2 \beta - i \sinh 2 \alpha$

Denominator $= 2\cos(\beta + i\alpha)\cos(\beta - i\alpha)$

$= 2 \cos A \cos B;\ A = \beta+i\alpha$

$B = \beta-i\alpha$

$= \cos (A+B) + \cos (A-B)$

$= \cos 2 \beta + \cos (i2 \alpha)$

$= \cos 2 \beta + \cos h2 \alpha$

Using in (1)

$$\tan h\ (\alpha+i\beta) = i\frac{\sin 2\beta - i\sinh 2\alpha}{\cos 2\beta + \cosh 2\alpha}$$

$$= \frac{i\sin 2\beta + \sinh 2\alpha}{\cos 2\beta + \cosh 2\alpha}$$

$$= \frac{\sinh 2\alpha}{\cos 2\beta + \cosh 2\alpha} + i\frac{\sin 2\beta}{\cos 2\beta + \cosh 2\alpha}$$

$$\text{Real part} = \frac{\sinh 2\alpha}{\cos 2\beta + \cosh 2\alpha}$$

$$\text{Imaginary part} = \frac{\sin 2\beta}{\cos 2\beta + \cosh 2\alpha}$$

Examples:

1. Prove that sinh3x = 3 sin hx + 4 sin h^3x

Proof : $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Put $\theta = ix$

Sin (3ix) = 3 sin(ix) – 4 [sin (ix)]^3

i sinh3x = i3 sinhx – 4 [$i^3\sinh^3 x$]

= i 3sinhx – 4 i^3 $\sinh^3$x

i sinh3x = 3 i sinhx + 4 i $\sinh^3$x

/ i; sinh3x = 3sinhx + 4 $\sinh^3$x

2. Express $\sinh^7\theta$ interms of hyperbolic sines of multiples of θ

Solution

$$\text{Sin h}\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

$$2\sinh\theta = e^{\theta} - e^{-\theta}$$

$$(2\sinh\theta)^7 = (e^{\theta} - e^{-\theta})^7$$

$2^7\sinh^7\theta = e^{7\theta} - 7c_1 e^{6\theta}\ e^{-\theta} + 7c_2 e^{5\theta}\ e^{-2\theta} - 7c_3 e^{4\theta}\ e^{-3\theta} + 7c_4 e^{3\theta}\ e^{-4\theta} - 7c_5$ $e^{2\theta}\ e^{-5\theta} + 7c_6 e^{\theta}\ e^{-\theta\ 6} - 7c_7 e^{-7\theta}$

$$= e^{7\theta} - e^{-7\theta} - 7e^{5\theta} + \frac{7.6}{1.2}e^{3\theta} - \frac{7.6.5}{1.2.3}e^{\theta} + \frac{7.6.5.4}{1.2.3.4}e^{-\theta} - \frac{7.6}{1.2}e^{-3\theta} + 7e^{-5\theta}$$

$$=(e^{7\theta} - e^{-7\theta} -) - 7e^{5\theta} + 21e^{3\theta} - 35e^{\theta} + 35e^{-\theta} - 21e^{-3\theta} + 7e^{-5\theta}$$

$$=(e^{7\theta} - e^{-7\theta}) - 7(e^{5\theta} - e^{-5\theta}) + 21(e^{3\theta} - e^{-3\theta}) +- 35(e^{\theta} - e^{-\theta})$$

÷ 2;

$$2^6 \sinh 7\theta \;=\; \frac{e^{7\theta} - e^{-7\theta}}{2} 7\left(\frac{e^{5\theta} - e^{-5\theta}}{2}\right) + 21\left(\frac{e^{3\theta} - e^{-3\theta}}{2}\right) + -35\left(\frac{e^{\theta} - e^{-\theta}}{2}\right)$$

$$= \sinh 7\theta \;-\; 7\sinh 5\theta \;+\; 21\sinh 3\theta \;-\; 35\sinh\theta$$

$$\text{Sinh}7\theta \;=\; \frac{1}{2^6}\left[\sinh 7\theta - 7\sinh 5\theta \;+ 21\sinh 3\theta \;- 35\sinh\theta\;\right]$$

3. If sin (θ + iϕ) = tan α+i sec α , prove that cos2θ cosh2ϕ = 3

Solution

Sin (θ +iϕ) = sin θ cos (iθ) + cos θ sin (iθ)

= sinθ coshθ + cosθ (i sinhθ)

= sin θ coshθ + i cosθ - sinhθ

Sin θ coshθ + i cos sinhθ = tan α + i sec α

Equate real and imaginary parts

Tan α = sin θ coshθ

Sec α = cos θ sinhθ

We know

$$\sec^2 \alpha - \tan 2\,\alpha = 1$$

$$\cos^2\theta \;\sinh^2\theta \;-\sin^2\theta \;\cosh^2\theta \;= 1$$

$$\left[\frac{1+\cos 2\theta}{2}\right]\left[\frac{\cosh 2\phi - 1}{2}\right] - \left[\frac{1-\cos 2\theta}{2}\right]\left[\frac{\cosh 2\phi + 1}{2}\right] = 1$$

(1+cos2θ) (cosh2ϕ -1) - (1-cos2θ) (cosh2ϕ +1) = 4

On simplification

Cos2θ cosh2ϕ = 3

4. If sin (x+iy) = cosθ + i sinθ , prove that

$\cos^2 x = \sinh^2 y$

Proof : sin (x+iy) = cosθ + i sinθ

sin x cos (iy) + cos x sin (iy) = cos θ + i sin θ

sin x coshy + cos x (i sinhy) = cosθ + i sin θ

sin x coshy + i cosx (i sinhy) = cosθ + i sin θ

equate real and imaginary parts

sin x coshy = cos θ

cos x sinhy = sin θ

we know $\cos^2 x + \sin^2 \theta = 1$

$\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y = 1$

$(1 - \cos^2 x)(1+\sinh^2 y) + \cos^2 x \sinh^2 y = 1$

$1 + \sinh^2 y - \cos^2 x - \cos^2 x \sinh^2 y + \cos^2 x \sinh^2 y = 1$

$\cos^2 x = \sinh^2 y$

5. If sin (x+iy) = tan (α+iβ), show that

Sin2α tanhy = sin h2β tan x

Solution :

sin (x+iy) = sin x coshy + i cosx sinhy

$$\tan(\alpha+i\beta) = \frac{\mathrm{Sin}2\alpha}{\cos2\alpha+\cosh2\beta} + \frac{\mathrm{Sinh}2\beta}{\cos2\alpha+\cosh2\beta}$$

$$\sin x \cosh y = \frac{\mathrm{Sin}2\alpha}{\cos2\alpha+\cosh2\beta} \quad \text{-------------- (1)}$$

$$\cos x \sinh y = \frac{\mathrm{Sinh}2\beta}{\cos2\alpha+\cosh2\beta} \quad \text{-------------- (2)}$$

(1) ÷ (2)

$$\tan x \coth y = \frac{\mathrm{Sin}2\alpha}{\mathrm{Sinh}2\beta}$$

$$\tan x \sinh 2\beta = \sin 2\alpha \tanh y$$

2.3. Let us Sum up

So far we have studied the concept on finding the real and imaginary parts of trigonometric fuction using cos(ix)=coshx: sin(ix)=isinhx

2.4 Check your progress

(i) Prove that $\cosh^2 x = 1 + \sinh^2 x$

(ii) Prove that $\sinh 2x = 2\sinh x \cosh x$

2.5 Lesson end activities

1. Prove that $\cosh 3x = 4 \cosh 3x - 3\cosh x$
2. Express $\cosh 6\theta$ in a series of hyperbolic cosines of multiples of θ
3. If $\tan(\theta + i\theta) = \text{Sin}(x+iy)$, prove that $\coth y \sinh 2y = \cot x \sin 2\theta$
4. If $\sin(\theta + i\theta) = R(\cos\alpha + i\sin\alpha)$,

   Prove that

   $2R^2 = \cosh 2\phi - \cos 2\theta$

   And $\tan\alpha \tan\theta = \tanh\phi$

2.6 Points for discussion

5. If $(\cos(a+ib) = \alpha + i\beta$, prove that
   - i) $(1+\alpha)^2 + \beta^2 = (\cosh b + \cos a)^2$
   - ii) $(1-\alpha)^2 + \beta^2 = (\cosh b - \cos a)^2$
6. If $\cos(A+iB) = \cos\theta + i\sin\theta$, prove that $\sin\theta = \pm \sin^2 A$
7. If $\cos(A+iB) = \cos\theta + i\sin\theta$, prove that $\sin\theta = \pm \sin^2 B$
8. If $\sin(A+iB) = \lambda + iy$, prove that

   (i) $\dfrac{\lambda^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 A} = 1$

   (ii) $\dfrac{\lambda^2}{\sin^2 A} + \dfrac{y^2}{\cos^2 A} = 1$

   Answer : ② : $\dfrac{1}{32}\left[\cosh 6\theta + 6\cosh 4\theta + 15\cosh 2\theta + 10\right]$

2.7 References

Trigonometry by Rasinghamic and Aggarval

Lesson - 3

Logarithm of a complex number

Contents

3.0 Aim and Objectives

In this lesson we are going to see the definition of logarithm of a complex number using the fundamental concepts of logarithm of a function.

Definition 1: Let $Z = x+iy$, if $\log z = u$, then

$$Z = e^u$$

In general the logarithm of a complex number is also a complex number

3.1 To find $\log_e (x+iy)$

Let $\log_e (x+iy) = \alpha+i\beta$

$$\begin{aligned} x+iy &= e^{\alpha+i\beta} \\ &= e^{\alpha} e^{i\beta} \\ &= e^{\alpha} (\cos\beta + i \sin\beta) \\ x+iy &= e^{\alpha} \cos\beta + i e^{\alpha} \sin\beta \end{aligned}$$

equate real, imaginary parts

$$x = e^{\alpha} \cos\beta \qquad (1)$$

$$y = e^{\alpha} \sin\beta \qquad (2)$$

$(1)^2 + (2)^2$ gives

$$x^2 + y^2 = e^{2\alpha} \cos^2\beta + e^{2\alpha} \sin^2\beta$$

$$= e^{2\alpha}(\cos^2\beta + \sin^2\beta)$$

$$x^2 + y^2 = e^{2\alpha}$$

$$2\alpha = \log_e(x^2 + y^2)$$

$$A = \tfrac{1}{2}\log(x^2 + y^2)$$

② ÷ ① gives $\frac{y}{x} = \tan\beta$

$$\beta = \tan^{-1}\left(y/x\right)$$

$$\therefore \quad \log_e(x+iy) = \frac{1}{2}\log_e(x^2+y^2) + i\tan^{-1}\left(y/x\right)$$

To find general logarithm of a complex number

Let $\log_e(x+iy) = \alpha + i\beta$

$$x+iy = e^{\alpha+i\beta}$$

$$= e^{\alpha}e^{i\beta}$$

$$= e^{\alpha}\cos\beta + i\sin\beta$$

$$x+iy = e^{\alpha}(\cos(2n\pi+\beta) + i\sin(2n\pi+\beta)) \qquad n=1,2,3,..$$

$$= e^{\alpha}e^{i(2n\pi+\beta)}$$

$$= e^{\alpha+2n\pi i+i\beta}$$

$$x+iy = e^{\alpha+i\beta+2n\pi i}$$

$$\therefore \quad \mathrm{Log}_e(x+iy) = \alpha + i\beta + 2n\pi i$$

$$\mathrm{Log}_e(x+iy) = \log_e(x+iy) + 2n\pi$$

This is called the general logarithm of x+iy.

$$\therefore \quad \mathrm{Log}(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\left(y/x\right) + 2n\pi i$$

$$= \frac{1}{2}\log(x^2+y^2) + i\left[\tan^{-1}\left(y/x\right) + 2n\pi\right]$$

Note :

1. log Z is infinitely many valued function. This is called the general logarithm of Z.
2. If n = 0, we get the principal value of log Z

Important

1. $\log(x+iy)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\left(y/x\right)$

2. $\log_e(x+iy)=\log(x+iy)+2n\pi i$

3.2 Examples

1. find log (1+i)

$1 + I = x + iy \qquad \therefore \qquad x=1, y=1$

$x^2+y^2 = 2$

amplitude = $\tan^{-1}\left(y/x\right)=\tan^{-1}(1)=i\ \pi/4$

$\therefore \qquad \log(x+iy)=\frac{1}{2}\log 2+i\pi/4$

2. Find Log (1+i)

Log (1+i) = log (1+i) + 2 nπ i

$=\frac{1}{2}\log 2+i\pi/4+i2n\pi$

$=\frac{1}{2}\log 2+i(\pi/4+2n\pi)$

3. Log i

$i = 0 + 1\,i \qquad \therefore \quad x=0, y=1$

$\theta = \tan^{-1}\left(y/x\right) = \tan^{-1}\left(1/0\right)=\tan^{-1}\alpha = \pi/2$

$\sqrt{x^2+y^2} = \sqrt{1}=1$

Log i = $\frac{1}{2}\log(x^2+y^2)+i\theta$

$= \frac{1}{2}\log 1+i\pi/2$

$\mathrm{Log}\ i = i\pi/2 \qquad \{\therefore \quad \log 1 = 0\}$

4. Log i = log i + 2nπi

$= i\pi/2 + 2n\pi i$

5. Prove that log (cosθ + i sin θ) = i θ , $-\pi < \theta < \pi$

Solution :

log (cos θ + i sin θ) = log (x+iy)

x = cos θ ; y = sin θ

$\sqrt{x^2 + y^2} = 1; \qquad \theta = \tan^{-1}\left(y/x\right)$

$$\therefore \quad \log(\cos\theta + i\sin\theta) = \frac{1}{2}\log 1 + i\theta$$

$$= i\theta$$

6. Find a power series for $\tan^{-1}x$ using logarithm of complex number

Proof: if x is real

$\log(1+ix) = \frac{1}{2}\log(1+x^2) + i\tan^{-1}x$

$\tan^{-1}x$ = Imaginary part of log (1+ix)

but log (1 + Z) = $Z - Z^3/2 + Z^3/3 \ldots\ldots\ldots\ldots$

$\tan^{-1}x = \text{IP of } ix - \frac{i^2x^2}{2} + \frac{i^3x^3}{3} + \ldots\ldots$

$= \text{IP of } ix + \frac{x^2}{2} - \frac{i^3x^3}{3} + \ldots\ldots$

$$= \text{IP of } \left[\left(\frac{x^2}{2} + \frac{x^4}{4}\ldots\ldots\right) + i\left(x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right)\right]$$

$$\therefore \quad \mathrm{t{=}a}^{-1}\,x\mathrm{n} - .x\ .\frac{x^3}{3} + .\frac{x^5}{5}\ .\ .\ .$$

7. Obtain the general value of $L\mathrm{o}\,{}^{i}_{i}$

Solution : Let $L\mathrm{o}\,{}^{i}_{i}$ = a + ib

$i = i^{a+ib}$

$= e^{(a+ib)} \log i$

$= e^{(a+ib)\,[i(2n\pi+\pi/2)]}$

$i = e^{-b(2n\pi+\pi/2)}\, e^{ia(2n\pi+\pi/2)}$

taking modulus on both sides

$1 = e^{-b(2n\pi+\pi/2)}$

$b = 0$

$i = e^{ia(2n\pi+\pi/2)}$

$i = \cos[a(2n\pi+\pi/2)] + i \sin[a(2n\pi+\pi/2)]$

$= \cos[a(2n\pi+\pi/2)] = 0$

$a(2n\pi+\pi/2) = 2m+\pi/2$

$$a\left(\frac{4n\pi+\pi}{2}\right) = \frac{4n\pi+\pi}{2}$$

$$a = \frac{4m+1}{4n+1}; \quad m, n \leftarrow Z$$

$$\therefore \quad \log_i^i = \frac{4m+1}{4n+1}$$

Method 2

$$Log_i^i = \frac{Log_e^i}{Log_e^i} \qquad (1)$$

$$Log_e^i = \ln_e^i + i2n\pi$$

$$= i\pi/2 + 2n\pi i$$

$$= i\left(\pi/2 + 2n\pi\right)$$

$$= i\left(\frac{\pi+4n\pi}{2}\right) n \in ; Z$$

Similarly $Log_e^i = i\left(\frac{4n\pi+\pi}{2}\right) m \in ; \quad Z$

Using in (1)

$$\log_i^i = \frac{i\left(\frac{4n\pi+\pi}{2}\right)}{i\left(\frac{4m\pi+\pi}{2}\right)}$$

$$= \frac{(4n+1)\pi}{(4m+1)\pi}; n, m \in Z$$

8. If i $^{x+iy}$ = x+iy, prove that

$$x^2 + y^2 = e^{-(4n+1)\pi y}$$

Proof : $i^{x+iy} = x+iy$

$$x+iy = i^{x+iy}$$

$$= e^{\log i^{x+iy}}$$

$$= e^{(x+iy)\log i}$$

$$x+iy = e^{(x+iy)(2n\pi i + i\frac{\pi}{2}}$$

$$x+iy = e^{i(x+iy)\frac{(4n\pi+\pi)}{2}}$$

$$= e^{\frac{1}{2}(ix-y)4n\pi+\pi}$$

$$= e^{-\frac{y}{2}(4n\pi+\pi)} \quad e^{ix/2(4n\pi+\pi)}$$

$$= e^{-\frac{y}{2}(4n+1)\pi}\left[\cos\left(\frac{x}{2}(4n+1)\pi\right) + i\sin\left(\frac{x}{2}(4n+1)\pi\right)\right]$$

$$x = e^{-\frac{y}{2}(4n+1)\pi} \cos\left(\frac{x}{2}(4n+1)\pi\right)$$

$$y = e^{-\frac{y}{2}(4n+1)\pi} \sin\left(\frac{x}{2}(4n+1)\pi\right)$$

$$\therefore \quad x^2 + y^2 = e^{-y(4n+1)\pi}$$

3.3 Let us sum up.

So far we have studied the concept of finding the logarithm of a complex number and also a here concept of the general logarithm a complex number.

3.4 Check you progress

(a) Find log (1-i)

(b) Find log (1+ i tan2)

3.5 Lesson End Activities

1. Prove that $i^i = e^{-(4n+1)\pi/2}$; n is any integer
2. Prove that Log (-1) = i(2n+1) π
3. Prove that Log (1-i) = $\frac{1}{2}\log 2 + i\left[2n\pi + \left(-\pi/4\right)\right]$
4. Prove that Log (-5) = log 5 + I (2n π+ π)
5. If = A+iB , prove that
    a. $\tan\left(\frac{\pi A}{2}\right) = \frac{B}{A}$ and
    b. $A^2 + B^2 = e^{-\pi B}$
6. Prove that $i\log\frac{x-i}{x+i} = \pi - 2\tan^{-1} x$

3.6 Points for discussion

7. Show that $\log\frac{a+ib}{a-ib} = 2i\tan^{-1}\left(b/a\right)$
8. Prove that $\log(1+i\tan\alpha) = \log\sec\alpha + i\alpha ; 0 < \alpha < \frac{\pi}{2}$
9. Prove that $\log(1+\cos 2\theta + i\sin 2\theta) = \log(2\cos\theta) + i\theta \ ** -\pi < \theta < \pi$

3.7 References

1. Trigonometry by S. Narayanan

Lesson - 4

Summation of Series

Contents

In this lesson, we are going to study trigonometric series using the concept of arithmetic series using the concept of arithmetic progression, Geometric progression, Binominal theorem, exponential Theorem and logarithmic theorem.

Model 1 :

Summation of series when angles are in Arithmetic progression

Sine series

1.4.1 Find the sum to n terms of the series

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \sin(\alpha+3\beta) + \ldots\ldots$$

Proof :

Let $s_n = \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots. + \sin(\alpha+\overline{n-1}\beta)$

Multiply both sides by 2 sin $\left(\beta/2\right)$

$$2\sin\left(\beta/2\right)s_n = 2\sin\alpha\sin\beta/2 + 2\sin\beta/2\sin(\alpha+\beta) + 2\sin(\alpha+2\beta)\sin\beta/2 + \ldots + 2\sin(\alpha+\overline{n-1}\beta)\sin\beta/2 \qquad \text{------(1)}$$

We know 2 sin A sin B = cos (A-B) – cos (A+B)

$$\therefore \quad 2\sin\alpha \sin {}^{\beta}\!/_{2} = \cos\left(\alpha - {}^{\beta}\!/_{2}\right) - \cos\left(\alpha + {}^{\beta}\!/_{2}\right)$$

$$2\sin(\alpha+\beta)\sin {}^{\beta}\!/_{2} = \cos\left(\alpha + {}^{\beta}\!/_{2}\right) - \cos\left(\alpha + {}^{3\beta}\!/_{2}\right)$$

$$2\sin(\alpha+2\beta)\sin {}^{\beta}\!/_{2} = \cos\left(\alpha + {}^{3\beta}\!/_{2}\right) - \cos\left(\alpha + {}^{5\beta}\!/_{2}\right)$$

$$2\sin(\alpha + \overline{n-1}2\beta)\sin {}^{\beta}\!/_{2} = \cos\left(\alpha + \left(n - {}^{3}\!/_{2}\right)\beta\right) - \cos\left(\alpha + \left(n - {}^{1}\!/_{2}\right)\beta\right)$$

Adding the above we get

$$2\sin {}^{\beta}\!/_{2}\, S_n = \cos\left(\alpha - {}^{\beta}\!/_{2}\right) - \cos\left(\alpha + \left(n - {}^{1}\!/_{2}\right)\beta\right) \qquad \text{---(2)}$$

But we know that

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$A = \alpha - {}^{\beta}\!/_{2}\,;\, B = \alpha + \left(n - {}^{1}\!/_{2}\right)\beta$$

$$\therefore \quad A + B = \alpha - {}^{\beta}\!/_{2} + \alpha + \left(n - {}^{1}\!/_{2}\right)\beta$$

$$= 2\alpha - {}^{\beta}\!/_{2} + n\beta - {}^{\beta}\!/_{2}$$

$$= 2\alpha + n\beta - \beta$$

$$= 2\alpha + (n-1)$$

$$\therefore \quad \frac{A+B}{2} = \alpha + \frac{n-1}{2}\beta$$

$$A - B = \alpha - {}^{\beta}\!/_{2} - \alpha - \left(n - {}^{1}\!/_{2}\right)\beta$$

$$= \alpha - {}^{\beta}\!/_{2} - \alpha - n\beta + {}^{\beta}\!/_{2}$$

$$= -n\beta$$

$$\therefore \quad \frac{A-B}{2} = -n\frac{\beta}{2}$$

(2) becomes

$$2\sin {}^{\beta}\!/_{2}\, S_n = -2\sin\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{-n\beta}{2}\right)$$

$$= 2\sin\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{n\beta}{2}\right)$$

$$\therefore \qquad S_n = \frac{\sin\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\left({}^{\beta}\!/_{2}\right)}$$

Results

1. But $\beta = \alpha$

$$\therefore \qquad \sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots\ldots + \sin n\alpha$$

$$= \frac{\sin\left(\left(\frac{n+1}{2}\right)\alpha\right)\sin\left(\frac{n\alpha}{2}\right)}{\sin\left({}^{\alpha}\!/_{2}\right)}$$

4.2. Examples

Find the sum to n terms of the series

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cos(\alpha+3\beta) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Solution

$$S_n = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \ldots\ldots + \cos(\alpha + \overline{n-1}\beta)$$

Multiply both sides by $2\sin {}^{\beta}\!/_{2}$

$$\therefore 2\sin {}^{\beta}\!/_{2}\, S_n = 2\cos\alpha \sin {}^{\beta}\!/_{2} + 2\cos(\alpha+\beta)\sin {}^{\beta}\!/_{2} + 2\cos(\alpha+2\beta)\sin {}^{\beta}\!/_{2} + \ldots\ldots\ldots\ldots\ldots\ldots + 2\cos(\alpha + \overline{n-1}\beta)\sin {}^{\beta}\!/_{2} \quad \text{--(1)}$$

We know that

2 cos A sin B=sin (A+B) – sin (A-B)

$$2\cos\alpha \sin {}^{\beta}\!/_{2} = \sin(\alpha + {}^{\beta}\!/_{2}) - \sin(\alpha - {}^{\beta}\!/_{2})$$

$$2\cos(\alpha + {}^{\beta}\!/_{2})\sin {}^{\beta}\!/_{2} = \sin(\alpha + {}^{3\beta}\!/_{2}) - \sin(\alpha + {}^{\beta}\!/_{2})$$

$$2\cos(\alpha+2\beta)\sin{}^{\beta}\!/_{2} = \sin(\alpha+{}^{5\beta}\!/_{2}) - \sin(\alpha+{}^{3\beta}\!/_{2})$$

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$$2\cos(\alpha+\overline{n-1}\beta)\sin{}^{\beta}\!/_{2} = \sin(\alpha+(n-1/2)\beta) - \sin(\alpha+(n-3/2)\beta)$$

∴ using in (1)

$$2\sin{}^{\beta}\!/_{2}\, S_n = \sin\left[\alpha+\left(n-\frac{1}{2}\right)\beta\right] - \sin\left(\alpha-{}^{\beta}\!/_{2}\right) \qquad \text{----(2)}$$

But we know that

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

Here $A=\alpha+\left(n-{}^{1}\!/_{2}\right)\beta;\quad B=\alpha-{}^{\beta}\!/_{2}$

$$A+B=\alpha+\left(n-{}^{1}\!/_{2}\right)\beta+\alpha-{}^{\beta}\!/_{2}$$

$$=\alpha+n\beta-{}^{\beta}\!/_{2}+\alpha-{}^{\beta}\!/_{2}$$

$$=2\alpha+n\beta-\beta$$

$$=2\alpha+(n-1)\beta$$

$$\therefore\quad \frac{A+B}{2}=\alpha+\frac{n-1}{2}\beta$$

$$A-B=\alpha+n\beta-\frac{1}{2}\beta-\alpha+{}^{\beta}\!/_{2}$$

$$=n\beta$$

$$\therefore\quad \frac{A-B}{2}=\frac{n\beta}{2}$$

∴ using in (2)

$$2\sin{}^{\beta}\!/_{2}\, S_n = 2\cos\left[\alpha+\left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right)$$

$$\therefore \quad S_n = \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\beta/2\right)}$$

Cor 1: Put $\beta = \alpha$

$$\therefore \quad \cos\alpha + \cos 2\alpha + \cos 3\alpha + \ldots + \cos n\alpha = \frac{\cos\left(\frac{n+1}{2}\right)\alpha \sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\alpha/2\right)}$$

1. First the sum to n terms of the series

$\text{Sin}^2\alpha + \sin^2 2\alpha + \sin^2 3\alpha + \ldots\ldots\ldots\ldots\ldots..$

Solution

$S_n = \text{Sin}^2\alpha + \sin^2 2\alpha + \sin^2 3\alpha + \ldots\ldots\ldots\ldots + \sin^2 n\alpha$

$= ½ [2\ \text{Sin}^2\alpha + 2\sin^2 2\alpha + 2\sin^2 3\alpha + \ldots\ldots\ldots\ldots + 2\sin^2 n\alpha\ ]$

But we know $2\sin^2\alpha = 1 - \cos 2\alpha$

$$\therefore \quad S_n = \frac{1}{2}\left[(1-\cos 2\alpha) + (1-\cos 4\alpha) + (1-\cos 6\alpha) + \ldots\ldots + (1-\cos 2n\alpha)\right]$$

$$= \frac{1}{2}\left[n - (\cos 2\alpha + \cos 4\alpha + \cos 6\alpha + \ldots\ldots + \cos 2n\alpha)\right]$$

We know

$$\cos\alpha + \cos(\alpha + 2\beta) + \ldots.. + \cos(\alpha + \overline{n-1}\beta) = \frac{\cos\left[\alpha + \left(\frac{n-1}{2}\right)\beta\right]\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\beta/2\right)}$$

Hence 'α'$=2\alpha$; $\beta = 2\alpha$

$$\therefore \quad S_n = \frac{1}{2}\left[n - \frac{\cos\left[2\alpha + \frac{n-1}{2}2\alpha\right]\sin\left(\frac{n-2\alpha}{2}\right)}{\sin {}^{2\alpha}\!/_{2}} \right]$$

$$= \frac{1}{2}\left[n - \frac{\cos(n+1)\alpha \sin(n\alpha)}{\sin\alpha} \right]$$

2. Find the sum to n terms of the series

$\cos^2\alpha + \cos^2 2\alpha + \cos^2 2\alpha + \ldots\ldots\ldots\ldots + \cos^2 n\alpha + \ldots\ldots\alpha$

Proof:

Let

$S_n = \cos^2\alpha + \cos^2 2\alpha + \cos^2 2\alpha + \ldots\ldots\ldots\ldots + \cos^2 n\alpha$

$= ½ [2 \cos^2\alpha + 2 \cos^2 2\alpha + 2 \cos^2 2\alpha + \ldots\ldots\ldots\ldots + 2 \cos^2 n\alpha]$

But $1 + \cos 2\theta = 2\cos^2\theta$

$$\therefore \quad S_n = \frac{1}{2}\left[(1+\cos 2\alpha) + (1+\cos 4\alpha) + (1+\cos 6\alpha) + \ldots\ldots + (1+\cos 2n\alpha)\right]$$

$$= \frac{1}{2}\left[n + (\cos 2\alpha + \cos 4\alpha + \cos 6\alpha + \ldots\ldots + \cos 2n\alpha)\right]$$

$$= \frac{1}{2}\left[n + \frac{\cos\left[2\alpha + \frac{n-1}{2}2\alpha\right]\sin\left(\frac{n-2\alpha}{2}\right)}{\sin {}^{2\alpha}\!/_{2}} \right]$$

$$= \frac{1}{2}\left[n + \frac{\cos(n+1)\alpha \sin(n\alpha)}{\sin\alpha} \right]$$

3. Find the sum of the series

$\text{Sin}^3\alpha + \sin^3 2\alpha + \sin^3 3\alpha + \ldots\ldots\ldots\ldots\ldots\ldots + \sin^3 n\alpha$

Proof:

We know $\sin^3\alpha = 3\sin\alpha - 4\sin^3\alpha$

$\therefore \quad 4\sin^3\alpha = 3\sin\alpha - \sin 3\alpha$

Let $S_n = \mathrm{Sin}^3\alpha + \sin^3 2\alpha + \sin^3 3\alpha + \ldots\ldots\ldots\ldots + \sin^3 n\alpha$

$= \tfrac{1}{4}\,[4\,\mathrm{Sin}^3\alpha + 4\sin^3 2\alpha + 4\sin^3 3\alpha + \ldots\ldots\ldots\ldots + 4\sin^3 n\alpha\,]$

$$= \frac{1}{4}\left[\begin{array}{r}(3\sin\alpha - \sin 3\alpha) + (3\sin 2\alpha - \sin 6\alpha) + (3\sin 3\alpha - \sin 9\alpha) \\ + \ldots\ldots\ldots + (3\sin n\alpha - \sin 3n\alpha\end{array}\right]$$

$$= \frac{1}{4}\left[3(\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots\ldots\ldots + \sin n\alpha)\right]$$

$$-\left[\sin 3\alpha + \sin 6\alpha + \sin 9\alpha + \ldots\ldots\ldots + \sin 3n\alpha\right]$$

$$= \frac{3}{4}\left[\sin\alpha + \sin 2\alpha + \sin 3\alpha + \ldots\ldots\ldots + \sin n\alpha\right]$$

$$-\frac{1}{4}\left[\sin 3\alpha + \sin 6\alpha + \sin 9\alpha + \ldots\ldots\ldots + \sin 3n\alpha\right]$$

$$= \frac{3}{4}\,\frac{\sin\left[\alpha + \frac{n-1}{2}\alpha\right]\sin\left(\frac{n}{2}\alpha\right)}{\sin\left(\alpha/2\right)} - \frac{-1}{4}\,\frac{\sin\left[3\alpha + \frac{n-1}{2}3\alpha\right]\sin\left(\frac{3n\alpha}{2}\right)}{\sin\left(3\alpha/2\right)}$$

$$= \frac{3}{4}\,\frac{\sin\left(\frac{n+1}{2}\right)\alpha\,\sin\left(\frac{n\alpha}{2}\right)}{\sin\left(3\alpha/2\right)} - \frac{\frac{-1}{4}\sin\left[\frac{(n+1)3\alpha}{2}\right]\sin\left(\frac{3n\alpha}{2}\right)}{\sin\left(3\alpha/2\right)}$$

4. Find the sum to n terms of

$\mathrm{Cos}^3\alpha + \cos^3 2\alpha + \cos^2 3\alpha + \ldots\ldots\ldots\ldots + \cos^3 n\alpha$

Proof:

We know $\cos 3\theta = 4\cos 3\theta - 3\cos\theta$

$\therefore \quad 4\cos^3\theta = \cos 3\theta + 3\cos\alpha$

Let $S_n = \cos^3\alpha + \cos^3 2\alpha + \cos^3 3\alpha + \ldots\ldots\ldots\ldots + \cos^3 n\alpha$

$= \tfrac{1}{4}\,[4\cos^3\alpha + 4\cos^3 2\alpha + 4\cos^3 3\alpha + \ldots\ldots\ldots\ldots + 4\cos^3 n\alpha\,]$

$$= \frac{1}{4}\left[\begin{array}{r}(\cos 3\alpha + 3\cos\alpha) + (\cos 6\alpha + 3\cos 2\alpha) + (\cos 9\alpha - 3\cos 3\alpha) \\ + \ldots\ldots\ldots + (\cos 3n\alpha - 3\cos n\alpha\end{array}\right]$$

$$=\frac{1}{4}\left[\cos 3\alpha+\cos 6\alpha+\cos 9\alpha+.........+\cos 3n\alpha\right]+$$

$$3\left[\cos\alpha+\cos 2\alpha+\cos 3\alpha+.........+\cos\alpha\right]$$

$$=\frac{1}{4}\left[\cos 3\alpha+\cos 6\alpha+\cos 9\alpha+.........+\cos 3n\alpha\right]$$

$$+\frac{3}{4}\left[\cos\alpha+\cos 2\alpha+\cos 3\alpha+.........+\cos n\alpha\right]$$

$$=\frac{1}{4}\frac{\cos\left[3\alpha+\frac{n-1}{2}3\alpha\right]\sin\left(\frac{3n\alpha}{2}\right)}{\sin\left(3\alpha/2\right)}+\frac{3}{4}\frac{\cos\left[\alpha+\frac{n-1}{2}\alpha\right]\sin\left(\frac{n}{2}\alpha\right)}{\sin\left(\alpha/2\right)}$$

$$=\frac{\frac{1}{4}\cos\frac{(n+3)\alpha}{2}\sin\left(\frac{3n\alpha}{2}\right)}{\sin\left(3\alpha/2\right)}+\frac{3}{4}\frac{\cos\left(\frac{(n+1)\alpha}{2}\right)\sin\left(\frac{n\alpha}{2}\right)}{\sin\left(\alpha/2\right)}$$

Model 3 C+is method)

Type 1: Problems based on e^x and e^{-x}

1. $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+......................\infty$

2. $e^{-x}=1-\frac{x}{1!}+\frac{x^2}{2!}-\frac{x^3}{3!}+......................\infty$

Problems

1. Find the sum to infinity of the series

$$\sin\alpha+x\sin(\alpha+\beta)+\frac{x^2}{2!}\sin(\alpha+2\beta)+......................\infty$$

Proof:

$$\text{Let } S=\sin\alpha+x\sin(\alpha+i\beta)+\frac{x^2}{2!}\sin(\alpha+2\beta)+......................\infty$$

$$C=\cos\alpha+x\cos(\alpha+i\beta)+\frac{x^2}{2!}\cos(\alpha+2\beta)+......................\infty$$

$$C+iS=(\cos\alpha+i\sin\alpha)+x\cos(\alpha+\beta)+ix\sin(\alpha+\beta)+\frac{x^2}{2!}\cos(\alpha+2\beta)+$$

$$i\frac{x^2}{2!}\cos(\alpha+2\beta)....................\infty$$

$$=e^{i\alpha}+xe^{i(\alpha+\beta)}+\frac{x^2}{2!}e^{i(\alpha+2\beta)}+....................\infty$$

$$=e^{i\alpha}+x\,e^{i\alpha}e^{i\beta}+\frac{x^2}{2!}e^{i\alpha}e^{i2\beta}+....................\infty$$

$$=e^{i\alpha}\left[1+\frac{xe^{i\beta}}{1!}+\frac{x^2}{2!}e^{i2\beta}+....................\infty\right]$$

$$=e^{i\alpha}\left[1+\frac{xe^{i\beta}}{1!}+\frac{1}{2!}\left(xe^{i\beta}\right)^2+....................\infty\right]$$

$$=e^{i\alpha}\left[e^{xe^{i\beta}}\right]$$

$$=e^{i\alpha}e^{x(\cos\beta+i\sin\beta)}$$

$$=e^{i\alpha}e^{x\cos\beta+ix\sin\beta}$$

$$=e^{x\cos\beta}e^{i\alpha}e^{ix\sin\beta}$$

$$=e^{x\cos\beta}e^{i(\alpha+x\sin\beta)}$$

$$=e^{x\cos\beta}\left[\cos(\alpha+x\sin\beta)+i\sin(\alpha+x\sin\beta)\right]$$

$$C+is=e^{x\cos\beta}\cos(\alpha+x\sin\beta)+ie^{x\cos\beta}\sin(\alpha+x\sin\beta)$$

Equate imaginary part

$$S=e^{x\cos\beta}\sin(\alpha+x\sin\beta)$$

2. Find the sum of the series

$$\sin\alpha+\frac{1}{2!}\sin 2\alpha+\frac{1}{3!}+\sin 3\alpha+..........\ ....\ \infty$$

Proof

$$S=\sin\alpha+\frac{1}{2!}\sin 2\alpha+\frac{1}{3!}+\sin 3\alpha+.................\infty$$

Let

$$C = 1 + \cos\alpha + \frac{1}{2!}\cos 2\alpha + \frac{1}{3!} + \cos 3\alpha + \ldots\ldots\ldots\ldots\ldots\ldots\infty$$

$$\therefore \quad C + is = 1 + (\cos\alpha + i\sin\alpha) + \frac{1}{2!}(\cos 2\alpha + i\sin 2\alpha) + \frac{1}{3!} + (\cos 3\alpha + i\sin 3\alpha) + \ldots\ldots\ldots\ldots\ldots\ldots\infty$$

$$= 1 + e^{i\alpha} + \frac{1}{2!}e^{i2\alpha} + \frac{1}{3!}e^{i3\alpha} + \ldots\ldots\ldots\ldots\ldots\ldots\infty$$

$$= 1 + e^{i\alpha} + \frac{1}{2!}\left(e^{i\alpha}\right)^2 + \frac{1}{3!}\left(e^{i\alpha}\right)^3 + \ldots\ldots\ldots\ldots\ldots\ldots\infty$$

$$= 1 + y + \frac{1}{2!}y^2 + \frac{1}{3!}y^3 + \ldots\ldots\ldots\ldots\ldots\ldots\infty \quad where\ y = e^{i\alpha}$$

$$= e^{y}$$

$$= e^{e^{i\alpha}}$$

$$= e^{\cos\alpha + i\sin\alpha}$$

$$= e^{\cos\alpha} e^{i\sin\alpha}$$

$$= e^{\cos\alpha}\left[\cos(\sin\alpha) + i\sin(\sin\alpha)\right]$$

$$C + is = e^{\cos\alpha}\cos(\sin\alpha) + ie^{\cos\alpha}\sin(\sin\alpha)$$

Equate imaginary parts

$$S = e^{\cos\alpha}\sin(\sin\alpha)$$

3. Sum to infinity the series

$$\cos\alpha + \frac{\sin\alpha\cos 2\alpha}{1!} + \frac{\sin^2\alpha\cos 3\alpha}{2!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty$$

Solution

$$C = \cos\alpha + \frac{\sin\alpha\cos 2\alpha}{1!} + \frac{\sin^2\alpha\cos 3\alpha}{2!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty$$

$$S = \sin\alpha + \frac{\sin\alpha\sin^2\alpha}{1!} + \frac{\sin^2\alpha\sin 3\alpha}{2!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty$$

$$\therefore \quad C + is = (\cos\alpha + i\sin\alpha) + \frac{\sin\alpha}{1!}(\cos 2\alpha + i\sin 2\alpha)$$

$$+\frac{\sin^2\alpha}{2!}(\cos 3\alpha + i\sin 3\alpha) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty$$

$$= e^{i\alpha} + \frac{\sin\alpha}{1!}e^{i2\alpha} + \frac{\sin^2\alpha}{2!}e^{i3\alpha} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty$$

$$= e^{i\alpha}\left[1 + \frac{\sin\alpha}{1!}e^{i\alpha} + \frac{\sin^2\alpha}{2!}e^{i2\alpha} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty\right]$$

$$= e^{i\alpha}\left[1 + \frac{\sin\alpha}{1!}e^{i\alpha} + \frac{(\sin\alpha\ e^{i\alpha})^2}{2!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty\right]$$

$$= e^{i\alpha}\left[1 + \frac{y}{1!} + \frac{y^2}{2!} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty\right] \qquad where\ y = \sin\alpha e^{i\alpha}$$

$$= e^{i\alpha}e^{y}$$

$$= e^{i\alpha}e^{\sin\alpha e^{i\alpha}}$$

$$= e^{i\alpha}e^{\sin\alpha(\cos\alpha + i\sin\alpha)}$$

$$= e^{i\alpha + \sin\alpha\cos\alpha + i\sin^2\alpha}$$

$$= e^{\sin\alpha\cos\alpha}e^{i(\alpha + \sin^2\alpha)}$$

$$= e^{\sin\alpha\cos\alpha}\left[\cos(\alpha + \sin^2\alpha) + i\sin(\alpha + \sin^2\alpha)\right]$$

$$= e^{\sin\alpha\cos\alpha}\cos(\alpha + \sin^2\alpha) + ie^{\sin\alpha\cos\alpha}\sin(\alpha + \sin^2\alpha)$$

Equate real part

$$C = e^{\sin\alpha\cos\alpha}\cos(\alpha + \sin^2\alpha)$$

4. Sum the series to infinity

$$1 + \frac{\cos 2\theta}{2!} + \frac{\cos 4\theta}{2!} + \ldots\ldots\ldots\ldots\ldots\ldots\infty$$

Let $\quad C = 1 + \frac{\cos 2\theta}{2!} + \frac{\cos 4\theta}{2!} + \ldots\ldots\ldots\ldots\ldots\ldots\infty$

$$S = \frac{\sin 2\theta}{2!} + \frac{\sin 4\theta}{2!} + ...\infty$$

$$C + is = 1 + \frac{1}{2!}\left[\cos 2\theta + i\sin 2\theta\right] + \frac{1}{4!}\left[\cos 4\theta + \sin 4\theta\right] + ...\infty$$

$$= 1 + \frac{1}{2!}e^{i2\theta} + \frac{1}{4!}e^{i4\theta} + ...\infty$$

$$= 1 + \frac{1}{2!}\left(e^{i\theta}\right)^2 + \frac{1}{4!}\left(e^{i\theta}\right)^4 + ...\infty$$

$$= 1 + \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + ...\infty \qquad where\ x = e^{i\theta}$$

$$= \frac{e^x + e^{-x}}{2} = \cosh x$$

$$= \frac{1}{2}\left[e^{e^{i\theta}} + e^{-e^{i\theta}}\right] = \cosh(e^{i\theta})$$

$$= \cos(\cos \theta + i \sin \theta)$$

$$= \cos\left[i(\cos\theta + i\sin\theta)\right]$$

$$= \cos\left[i\cos\theta - \sin\theta)\right]$$

$$= \cos(i\cos\theta)\cos(\sin\theta) + \sin(i\cos\theta)\sin(\sin\theta)$$

$$= \cosh(\cos\theta)\cos(\sin\theta) + i\sinh(\cos\theta)\sin(\sin\theta)$$

Equate real part

$$C = \cosh(\cos\theta)\cos(\sin\theta)$$

Type 2 Summation of series based on logarithmic series

FORMULA

1. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$

2. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ....\infty$

(1) sum the series to infinity

$$a\cos\theta + \frac{a^2\cos 2\theta}{2} + \frac{a^3\cos 3\theta}{3} + ...\infty$$

Let $\quad C = a\cos\theta + \frac{a^2\cos 2\theta}{2} + \frac{a^3\cos 3\theta}{3} + ...\infty$

$$S = a\sin\theta + \frac{a^2\sin 2\theta}{2} + \frac{a^3\sin 3\theta}{3} + ...\infty$$

$$C + is = a(\cos\theta + i\sin\theta) + \frac{a^2}{2}(\cos 2\theta + i\sin 2\theta) +$$

$$\frac{a^3}{3}(\cos 3\theta + i\sin 3\theta) + ...\infty$$

$$= ae^{i\theta} + \frac{a^2}{2}ae^{i2\theta} + \frac{a^3}{3}ae^{i3\theta} + ...\infty$$

$$= x + \frac{x^2}{2} + \frac{x^3}{3} + ...\infty$$

Where $x = ae^{i\theta}$

$$= -\log(1-x)$$

$$= -\log(1-ae^{i\theta})$$

$$= -\log[1 - a(\cos\theta + i\sin\theta)]$$

$$= -\log[1 - a\cos\theta - isain\theta]$$

$$= -\left[\frac{1}{2}\log[1 - a\cos\theta)^2 + a^2\sin^2\theta] + i\tan^{-1}\left(\frac{-a\sin\theta}{1-a\cos\theta}\right)\right]$$

Equal to real part

$$C = -\frac{1}{2}\log\left[(1 - a\cos\theta)^2 + a^2\sin^2\theta\right]$$

$$= -\frac{1}{2}\log\left[1 + a^2\cos^2\theta + a^2\sin^2\theta - 2a\cos\theta\right]$$

$$= -\frac{1}{2}\log\left[1 + a^2 - 2a\cos\theta\right]$$

(2) sum the series to infinity

$$c\sin\alpha - \frac{1}{2}c^2\sin(\alpha+\beta) + \frac{1}{3}c^3\sin(\alpha+2\beta)..........\infty$$

Solution

$$S = c\sin\alpha - \frac{1}{2}c^2\sin(\alpha+\beta) + \frac{1}{3}c^3\sin(\alpha+2\beta)..........\infty$$

$$C = c\cos\alpha - \frac{1}{2}c^2\cos(\alpha+\beta) + \frac{1}{3}c^3\cos(\alpha+2\beta)..........\infty$$

$$\therefore C + is = c(\cos\alpha + i\sin\alpha) - \frac{1}{2}c^2\left[\cos(\alpha+\beta) + i\sin(\alpha+\beta)\right]$$

$$+\frac{1}{3}c^3\left[\cos(\alpha+2\beta) + i\sin(\alpha+2\beta)\right]$$

$$= c\,e^{i\alpha} - \frac{1}{2}c^2e^{i(\alpha+\beta)} + \frac{1}{3}c^3e^{i(\alpha+2\beta)}..............\infty$$

$$= c\,e^{i\alpha} - \frac{1}{2}c^2e^{i\alpha}e^{i\beta} + \frac{1}{3}c^3e^{i\alpha}e^{i2\beta}..............\infty$$

$$= e^{i\alpha}\left[c - \frac{1}{2}c^2e^{i\beta} + \frac{1}{3}c^3e^{i2\beta}..................\infty\right]$$

$$= \frac{e^{i\alpha}}{e^{i\beta}}\left[ce^{i\beta} + \frac{1}{2}c^2e^{i2\beta} + \frac{1}{3}c^3e^{i3\beta}.................\infty\right]$$

$$= e^{i(\alpha-\beta)}\left[x - \frac{1}{2}x^2 + \frac{1}{3}x^3.................\infty\right]$$

Where $x = ce^{i\beta}$

$$= e^{i(\alpha-\beta)} \times \log(1+x)$$

$$= e^{i(\alpha-\beta)}\log(1+ce^{i\beta})$$

$$= e^{i(\alpha-\beta)}\log\left[1 + c(\cos\beta + i\sin\beta)\right]$$

$$= e^{i(\alpha-\beta)}\log[1+c\cos\beta)+ic\sin\beta)]$$

$$= [\cos(\alpha-\beta)+i\sin(\alpha-\beta)]\left[\begin{array}{r}\frac{1}{2}\log\left((1+c\cos\beta)^2+c^2\sin^2\beta\right)+\\ i\tan^{-1}\left(\frac{c\sin\beta}{1+c\cos\beta}\right)\end{array}\right]$$

Equate imaginary part

$$S=\frac{1}{2}\sin(\alpha-\beta)\log\left[(1+c\cos\beta)^2+c^2\sin^2\beta\right]+\cos(\alpha-\beta)\tan^{-1}\left(\frac{c\sin\beta}{1+c\cos\beta}\right)$$

Type 3: Summation of series – using Binomial series

1. Sum the series

$$1-\frac{1}{2}\cos\theta+\frac{1.3}{2.4}\cos 2\theta-\frac{1.3.5}{2.4.6}\cos 3\theta+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty$$

Solution :

$$C=1-\frac{1}{2}\cos\theta+\frac{1.3}{2.4}\cos 2\theta-\frac{1.3.5}{2.4.6}\cos 3\theta+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty$$

$$S=-\frac{1}{2}\sin\theta+\frac{1.3}{2.4}\sin 2\theta-\frac{1.3.5}{2.4.6}\sin 3\theta+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty$$

$$C+is=1-\frac{1}{2}(\cos\theta+i\sin 2\theta)+\frac{1.3}{2.4}(\cos 2\theta+i\sin 2\theta)$$
$$-\frac{1.3.5}{2.4.6}(\cos 3\theta+i\sin 3\theta)+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty$$

$$=1-\frac{1}{2}e^{i\theta}+\frac{1.3}{1.2.2.2}e^{i2\theta}-\frac{1.3.5}{1.2.2.2.3.2}e^{i3\theta}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty$$

$$=1-\frac{1}{2}e^{i\theta}+\frac{1.3}{2!}\frac{e^{i2\theta}}{2^2}-\frac{1.3.5}{3!}\frac{e^{i3\theta}}{2^3}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty$$

$$=1-\frac{1}{1!}\left(\frac{e^{i\theta}}{2}\right)+\frac{1.3}{2!}\left(\frac{e^{i\theta}}{2}\right)^2-\frac{1.3.5}{3!}\left(\frac{e^{i\theta}}{2}\right)^3+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty$$

This is of the form

$$1-\frac{p}{1!}\left(x/q\right)+\frac{p(p+8)}{2!}\left(x/q\right)^2-\frac{p(p+q)(p+2q)}{3!}\left(\frac{x}{q}\right)^3+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty$$

$p = 1; \qquad p + q = 3 \qquad \therefore \quad q = 2$

$$\frac{x}{q} = \frac{e^{i\theta}}{2}$$

$$\frac{x}{2} = \frac{e^{i\theta}}{2}$$

$x = e^{i\theta}$

$$\therefore \qquad C + is = (1 + x)^{-p/q}$$

$$= (1 + e^{i\theta})^{-1/2}$$

$$C + is = (1 + \cos\theta + i\sin\theta)^{-1/2}$$

$$= \left[2\cos^2 \theta/2 + i2\sin\theta/2 \cos\theta/2\right]^{-1/2}$$

$$= \left(2\cos\theta/2\right)^{-1/2}\left[\cos\left(\theta/2\right) + i\sin\left(\theta/2\right)\right]^{-1/2}$$

$$= \left[2\cos\left(\theta/2\right)\right]^{-1/2}\left(e^{i\theta/2}\right)^{-1/2}$$

$$= \left[2\cos\left(\theta/2\right)\right]^{-1/2}\left(e^{i\theta/4}\right)$$

$$C + is = (2\cos\theta)\left[\cos\theta/4 - i\sin\theta/4\right]$$

Equate real part on both sides

$$C = (2\cos\theta)^{-1/2}\cos\left(\theta/4\right)$$

4.3 Let us sum up

So far we have studied the concept of finding a trigonometric series using ap, gp, etc. Also we have seen the sum to arrive a certain trigonometric series using the fundamental forms of sin3x, cos3x, $\sin^3 x$, $\cos^3 x$, $\sin^2 x$, $\cos^2 x$

4.4. Check you progress

(a) Find sum for cos2α+cos4α+cosbα+….α

(b) Find sum for sin2α+sin4α+sinbα+….α

4.5. Lesson end Activities

sum the series to infinity

1. $\cos\alpha + \frac{\cos\beta}{1!}\cos(\alpha+\beta) + \frac{\cos^2\beta}{2!}\cos(\alpha+2\beta) + \ldots\ldots\ldots\ldots\infty$

Ans: $e^{\cos^2\beta}\cos(\alpha+\sin\beta+\cos\beta)$

2. $\sin\alpha + \frac{\sin(\alpha+2\beta)}{2!} + \frac{\sin(\alpha+4\beta)}{4!}\ldots\ldots\ldots\ldots\infty$

Ans: $[\sin\alpha\cos(\cos\beta).\cosh(\sin\beta) - \cos\alpha\sin\beta\sinh(\sin\beta)]$

3. $\sin\theta\cos\theta + \frac{\sin 3\theta}{3!}\cos 3\theta + \frac{\sin 5\theta}{5!}\cos 5\theta + \ldots\ldots\ldots\ldots$

Ans: $\sinh(\sin\theta\cos\theta)\cos(\sin^2\theta)$

4. $1 + \frac{1}{2}\cos 2\alpha - \frac{1}{2.4}\cos 4\alpha + \frac{1.3}{2.4.6}\cos 6\alpha\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\infty$

Ans: $\sqrt{2\cos\alpha}\,\cos\left(\alpha/2\right)$

4.6 Points for discussion

5. $a\sin\theta + \frac{a^2}{2}\sin 2\theta + \frac{a^3}{3}\sin 3\theta + \ldots\ldots\ldots\ldots\infty$

Ans: $\tan^{-1}\left[\frac{a\sin\theta}{1-a\cos\theta}\right]$

6. $\cos\alpha\sin\alpha - \frac{1}{2}\cos^2\alpha\sin 2\alpha + \frac{1}{3}\cos^3\alpha\sin 3\alpha$

Ans: $\tan^{-1}\left[\frac{\sin\alpha\cos\alpha}{1+\cos^2\alpha}\right]$

4.7 Sources

1) Trigonometry	:	M.L. Khanna
2) Trigonometry	:	S. Narayanan

Unit II

Lesson - 5

Vector Calculus

Contents

5.0 Aims and Objectives

We are going to study the concept of Scalar and Vector point functions, differential vectors in detail.

5.1 Scalar and Vector point functions – differential vectors

5.1.1. Scalar point function

Let A be any subset of the set of real numbers. If to each element a of A, we associate by some rule a unique real number f(a), then this rule defines a scalar function of the scalar variable a. Here f(a) is a scalar quantity and therefore f is a scalar function.

5.1.2. Vector point function

Let A be any subset of the set of real numbers. If to each element a of A, we associate by some rule a unique vector $\vec{f}(a)$, then this rule defines a vector function of the scalar variable 'a'. Here $\vec{f}(a)$ is a vector quantity and $\vec{f}$ is a vector function.

5.1.3. Derivative of a vector function with respect to a scalar

Let $\vec{r} = \vec{f}(t)$ be a vector function of the scalar variable 't'.

If $\frac{d\vec{r}}{dt}$ exists, there $\vec{r}$ is said to be differentiable.

$\frac{d\vec{r}}{dt}$ is denoted by $\vec{r}$. Similarly $\frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}$........ are denoted by $\vec{r}, \vec{r}$....... respectively.

5.1.4. Some results on differentiation of vectors.

Let $\overleftarrow{a}, \vec{b}, \vec{c}$ be differentiable vector function of a scalar t. let ϕ be a differentiable scalar point function of the same variable 't' , then

a. $\frac{d}{dt}(\vec{a}+\vec{b}) = \frac{d\vec{a}}{dt} + \frac{d\vec{b}}{dt}$

b. $\frac{d}{dt}(\vec{a}.\vec{b}) = \vec{a}.\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}.\vec{b}$

c. $\frac{d}{dt}(\vec{a}\times\vec{b}) = \vec{a}\times\frac{d\vec{b}}{dt} + \frac{d\vec{a}}{dt}\times\vec{b}$

d. $\frac{d}{dt}(\phi\vec{a}) = \phi\frac{d\vec{a}}{dt} + \frac{d\phi}{dt}\vec{a}$

e. $\frac{d}{dt}\left[\vec{a}\vec{b}\vec{c}\right] = \left[\frac{d\vec{a}}{dt}\vec{b}\vec{c}\right] + \left[\vec{a}\frac{d\vec{b}}{dt}\vec{c}\right] + \left[\vec{a}\vec{b}\frac{d\vec{c}}{dt}\right]$

f. $\frac{d}{dt}\left[\vec{a}\times(\vec{b}\times\vec{c})\right] = \frac{d\vec{a}}{dt}\times(\vec{b}\times\vec{c}) + \vec{a}\times\left(\frac{d\vec{b}}{dt}\times\vec{c}\right) + \vec{a}\times\left(\vec{b}\times\frac{d\vec{c}}{dt}\right)$

5.1.5. Derivative of a constant vector.

A vector is said to be constant only if both its magnitude and direction are fixed.

Let $\vec{r}$ be a constant vector function of the scalar variable t

Let $\vec{r} = \vec{c}$; $\vec{c}$ is a constant vector

$\therefore \quad \vec{r} + \delta\vec{r} = \vec{c}$

$\therefore \quad \vec{r} + \delta\vec{r} - \vec{r} = \vec{0}$

$\delta\vec{r} = \vec{0}$

$$\therefore \quad \frac{\delta \vec{r}}{\delta t} = 0$$

$$\therefore \quad \lim_{\delta t \to 0} \frac{\delta \vec{r}}{\delta t} = \vec{0}$$

$$\therefore \quad \frac{\delta \vec{r}}{\delta t} = \vec{0} \quad \text{(zero vector)}$$

5.1.6 Derivative of a vector point function in terms of its components

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ where x, y, z are scalar functions of the sector variable ‘t’.

Then $\frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$

Results

(1) The necessary and sufficient condition for the vector point function $\vec{a}(t)$ to be constant is that $\frac{d\vec{a}}{dt} = \vec{0}$

(2) If $\vec{a}$ is a differentiable vector point function of the scalar variable ‘t’ and if $|\vec{a}| = a$, then

a. $\frac{d}{dt}\left(\vec{a}^{\,2}\right) = 2a\frac{da}{dt}$

b. $\vec{a}.\frac{d\vec{a}}{dt} = a\frac{da}{dt}$

c. if $\vec{a}$ has constant length, then $\vec{a}. \perp r\frac{d\vec{a}}{dt}, if \left|\frac{d\vec{a}}{dt}\right| = 0$

d. The necessary and sufficient condition for the vector $\vec{a}(t)$ to have constant magnitude is $\vec{a}.\frac{d\vec{a}}{dt} = 0$

5.2 Examples

1. If $\vec{a}$ is a differentiable vector function of the scalar variable ‘t’ , then

$$\frac{d}{dt}\left(\vec{a}\times\frac{d\vec{a}}{dt}\right)=\vec{a}\times\frac{d^2\vec{a}}{dt^2}$$

Proof :

$$\frac{d}{dt}\left(\vec{a}\times\frac{d\vec{a}}{dt}\right)=\frac{d\vec{a}}{dt}\times\frac{d\vec{a}}{dt}+\vec{a}\times\frac{d}{dt}\left(\frac{d\vec{a}}{dt}\right)$$

$$=\vec{0}+\vec{a}\times\frac{d^2\vec{a}}{dt^2}$$

$$=\vec{a}\times\frac{d^2\vec{a}}{dt^2} \qquad \left\{\because \vec{a}\times\vec{a}=\vec{0}\right\}$$

2. The necessary and sufficient condition for the vector $\vec{a}$ (t) to have a constant direction is $\vec{a}\times\frac{d\vec{a}}{dt}=\vec{0}$.

Proof: Let $\vec{a}$ be the unit vector in the direction of $\vec{a}$

$$\therefore \quad \vec{a}=\left|\vec{a}\right|\hat{a}$$

$$\therefore \quad \frac{d\vec{a}}{dt}=\left|\vec{a}\right|\frac{d\hat{a}}{dt}+\frac{d\left|\vec{a}\right|}{dt}\hat{a}$$

$$\therefore \quad \vec{a}\times\frac{d\vec{a}}{dt}=\left|\vec{a}\right|\hat{a}\times\left[\left|\vec{a}\right|\frac{d\hat{a}}{dt}+\frac{d\left|\vec{a}\right|}{dt}\hat{a}\right]$$

$$\left|\vec{a}\right|^2\hat{a}\times\frac{d\hat{a}}{dt}+\vec{0}$$

$$=\left|\vec{a}\right|^2\hat{a}\times\frac{d\hat{a}}{dt} \qquad (1)$$

<u>The condition is necessary</u>

Suppose $\vec{a}$ has a constant direction then $\hat{a}$ is a constant vector

$\therefore \quad \frac{d}{dt}(\hat{a}) = 0$

$\therefore$ using in (1) $\vec{a} \times \frac{d\vec{a}}{dt} = \vec{0}$

$\therefore$ The condition is necessary

The condition is sufficient

Suppose $\vec{a} \times \frac{d\vec{a}}{dt} = \vec{0}$

From (1), $\left|\vec{a}\right|^2 \hat{a} \times \frac{d\hat{a}}{dt} = \vec{0}$

$\therefore \quad \hat{a} \times \frac{d\hat{a}}{dt} = \vec{0}$ ---(2)

Since $\hat{a}$ is of constant length,

$\therefore \quad \hat{a}.\frac{d\hat{a}}{dt} = 0$ ---(3)

$\therefore$ $\hat{a}$ is a constant vector is the direction of $\vec{a}$ is constant

3. Find a unit tangent vector to the curve x = 3t+2; y=5t^2, z=2t-1 at t=1

Solution : Unit tangent vector is defined as $\frac{d\vec{r}}{dt} \div \left|\frac{d\vec{r}}{dt}\right|$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= (3t+2)\vec{i} + 5t^2\vec{j} + (2t-1)\vec{k}$$

$$\frac{d\vec{r}}{dt} = 3\vec{i} + 10t\vec{j} + 2\vec{k}$$

$$\therefore \quad \left.\frac{d\vec{r}}{dt}\right]_{t=1} = 3\vec{i} + 10t\vec{j} + 2\vec{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{9+100+4} = \sqrt{113}$$

$\therefore$ unit tangent vector $= \dfrac{3\vec{i} + 10t\vec{j} + 2\vec{k}}{\sqrt{113}}$

4. A particle moves along a curve whose parametric equations are x = e^{-t}; y = 2 cos3t, z = 2sin3t, where t is the time. Find its velocity and acceleration at t = 0.

Solution :

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= e^{-t}\vec{i} + 2\cos 3t + 2\sin 3t\vec{k}$$

$$\frac{d\vec{r}}{dt} = -e^{-t}\vec{i} - 6\sin 3t\vec{j} + 6\cos 3t\vec{k}$$

$$\frac{d^2\vec{r}}{dt} = e^{-t}\vec{i} - 18\cos 3t\vec{j} - 18\sin 3t\vec{k}$$

$\therefore$ At t = 0, $\dfrac{d\vec{r}}{dt} = -\vec{i} + 6\vec{k}$

Which is the required velocity vector

$\therefore$ At t = 0, $\dfrac{d^2\vec{r}}{dt^2} = \vec{i} - 18\vec{j}$

Which is the required acceleration vector.

5.3. Let us sum up

So far we have studied in finding the velocity, acceleration of a particle using fundamental of calculus.

5.4. Check your progress

(1) Find $\dfrac{d\vec{r}}{dt}, \dfrac{d^2\vec{r}}{dt^2}$ if

$\vec{r} = x\vec{\ }\, i + \vec{y} + 2\vec{\ }$ where

x=t, y=t^2, z=t^3 at t =1

(2) Find $\left|\dfrac{d\vec{\ }}{d}\right|\begin{matrix}x\\ i\\ t\end{matrix}\ \vec{f} = rc\ \vec{a} + si\ s\ \vec{b}\ ı$ or t=0

5.5 Lesson End Activities

1. If $\vec{r} = \vec{a}$ cos nt + $\vec{b}$ sin nt where $\vec{a},\vec{b}$,n are constants, prove that

a. $\vec{r} = \frac{d\vec{r}}{dt} = n\vec{a} \times \vec{b}$

b. $\frac{d^2\vec{r}}{dt^2} + n^2\vec{r} = 0$

c. $\left[\vec{r}\,\frac{d\vec{r}}{dt}\,\frac{d^2\vec{r}}{dt^2}\right] = 0$

2. Find the unit tangent vector to the curve x = a cost; y = a sint; z = ct

Ans $\frac{a\sin t\vec{i} + a\cos t\vec{y} + c\vec{k}}{\sqrt{a^2 + c^2}}$

3. Find the velocity vector, the speed and the acceleration vector for the particle whose path is given by

x = 3 cos2t; z = 2 sin3t

Ans:

i) Velocity = $-6\sin 2t\vec{i} + 6\cos 3t\vec{k}$

ii) Speed = $\sqrt{36(\cos^2 3t + \sin^2 2t}$

iii) Acceleration = $-12\cos 2t\vec{i} - 18\sin 3t\vec{k}$

5.6 Points for discussion

4. The position vector of a moving point as given by $\vec{r} = \cos at\vec{i} + \sin at\vec{j}$. Show that the velocity $\vec{v}$ is perpendicular to $\vec{r}$

5. If $\vec{r} = \cos at\vec{i} + \sin at\vec{j}$, show that $\vec{r} \times \vec{v}$ is a constant vector.

5.7 References

1) Vector Calculus by - Namasivayam
2) Vector Calculus by - Rasinghmia aggarval
3) Vector calculus by - P. Durai Pandian
4) Vector calculus by - Chatterjee

Lesson - 6

Contents

6.0 Aims and Objectives

We are going to study the concept of new ideas on del operator. We also study the concept of operators line dir $\overline{f}$, curl and grade $\overline{f}$. Also our aim is to study the concept of solenoidal vectors and irrotational vectors.

6.1 Differential operators and directional derivative

6.1.1 The vector differential operator ∇ (del) is defined as

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

6.1.2 The gradient

Let $\phi(x, y, z)$ be a differentiable scalar field. Then gradient of ϕ is defined as

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} \text{ and is denoted by grad } \phi$$

$$\therefore \quad grad\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)\phi$$

$$= \nabla\phi$$

$\nabla\phi$ is a vector field.

6.1.3. Divergence of a vector point function Let $\vec{f}(x,y,z)$ be a vector point function differentiable at each point (x,y,z) in a certain region of space. The divergence of $\vec{f}$ is defined as

$$\vec{i}.\frac{\partial \vec{f}}{\partial x}+\vec{j}.\frac{\partial \vec{f}}{\partial y}+\vec{k}.\frac{\partial \vec{f}}{\partial z} \text{ and is}$$

Written as div $\vec{f}$

$$\therefore\ \operatorname{div}\vec{f} = \vec{i}.\frac{\partial \vec{f}}{\partial x}+\vec{j}.\frac{\partial \vec{f}}{\partial y}+\vec{k}.\frac{\partial \vec{f}}{\partial z}$$

$$=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right).\vec{f}$$

$$=\nabla.\vec{f}$$

6.1.4. Solenoidal vector

A vector point function is called solenoidal if $\nabla.\vec{f}=0$

6.1.5 Curl of a vector point function.

Let $\vec{f}(x,y,z)$ be a differentiable vector point function in a certain region of space. Then the curl of $\vec{f}$ is defined as $\vec{i}\times\frac{\partial \vec{f}}{\partial x}+\vec{j}\times\frac{\partial \vec{f}}{\partial y}+\vec{k}\times\frac{\partial \vec{f}}{\partial z}$ and is written as curl $\vec{f}$.

$$\therefore\ curl\,\vec{f}=\vec{i}\times\frac{\partial \vec{f}}{\partial x}+\vec{j}\times\frac{\partial \vec{f}}{\partial y}+\vec{k}\times\frac{\partial \vec{f}}{\partial z}$$

$$=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times\vec{f}$$

$$=\nabla\times\vec{f}$$

6.1.6 A vector point function $\vec{f}$ is called irrotational or rotation of $\vec{f}$ if $\nabla\times\vec{f}=\vec{0}$

6.1.7 $\nabla.\vec{f}$ and $\nabla\times\vec{f}$ in terms of component's

1. Let $\vec{f}=f_1\vec{i}+f_2\vec{j}+f_3\vec{k}$

$$\nabla.\vec{f}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$$

2. $\vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$

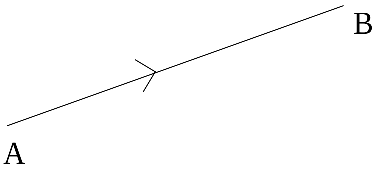
$$\nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial y}\right]\vec{i} - \vec{j}\left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right] + \vec{k}\left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]$$

6.1.8 Directional derivative

1. Let f be a scalar point function of the variable t.

Let f(A), f(B) be the functional values of



the scalar point function f at A and B respectively. Then if $\lim\limits_{B\to A} \frac{f(B)-f(A)}{AB}$ exists, it is called the directional derivative of the scalar functional A along AB.

2. Let $\vec{f}$ be a vector point function.

Then $\lim\limits_{B\to A} \frac{\vec{f}(B)-\vec{f}(A)}{AB}$ if it exists, is called the directional derivative of $\vec{f}$ at A along AB.

Note (1) : $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ are the directional derivatives of f at A in the directions of the coordinate axes

(2) $\frac{\partial \vec{f}}{\partial x}, \frac{\partial \vec{f}}{\partial y}, \frac{\partial \vec{f}}{\partial z}$ are the directional derivatives of $\vec{f}$ at A in the directional derivatives of $\vec{f}$ at A in the direction of the coordinate axes.

6.1.9 Directional derivative of f along any line

Let f be a scalar point function consider the line AB

Let be A (x,y,z)

Let the direction cosines of AB be l, m, n then the directional derivative of f along AB is defined as $l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z}$

6.1.10 Let $\vec{f}$ be a vector point function consider the line AB. Let A be (x,y,z)

Let the direction derivative of $\vec{f}$ along AB is defined as $l\frac{\partial \vec{f}}{\partial x} + m\frac{\partial \vec{f}}{\partial y} + n\frac{\partial \vec{f}}{\partial z}$

6.1.11. The directional derivative of ϕ in the direction of $\vec{n}$ is defined as $\frac{\nabla\phi.\vec{n}}{|\vec{n}|}$

a. If ϕ is a constant, then the directional derivative is zero.

b. $\nabla\phi$ is a vector normal to the level surface $\phi(x,y,z) = c$, c is a constant.

6.1.12. Level surface. Let f(x, y, z) be a scalar point function in a certain region of space. The set of all points of the region for which f becomes a constant is called a level surface and is written as f(x, y, z) = c, c is a constant.

a. The angle between the surfaces $\phi_1(x,y,z) = c_1$ *and* $\phi(x,y,z) = c_2$ is defined as the angle between their normal.

6.2 Example

Type 1

1. Find the directional derivative of $Z^2 + 2xy$ *at* $(1,-1,3)$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$

Solution :

$$\vec{n} = \vec{i} + 2\vec{j} + 2\vec{k}$$

$$|\vec{n}| = \sqrt{1+4+4} = 3$$

$$\phi(x,y,z) = Z^2 + 2xy$$

$$\frac{\partial f}{\partial x} = 2y; \frac{\partial f}{\partial y} = 2x; \frac{\partial f}{\partial z} = 2z$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= 2y\vec{i} + 2x\vec{j} + 2z\vec{k}$$

$$(1,-1,3) = (-2)\vec{i} + 2\vec{j} + 6\vec{k}$$

$$= -2\vec{i} + 2\vec{j} + 6\vec{k}$$

$$\nabla\phi.\vec{n} = \left(-2\vec{i} + 2\vec{j} + 6\vec{k}\right).\left(\vec{i} + 2\vec{j} + 2\vec{k}\right)$$

$$= -2 + 4 + 12$$

$$= 14$$

The directional derivative of ϕ along $\vec{n}$

$$= \frac{\nabla\phi.\vec{n}}{\left|\vec{n}\right|}$$

$$= \frac{14}{3}$$

2. Find the maximum directional derivative of $\phi = xyz^2$ at $(1,0,3)$

Solution:

$$\phi = xyz^2$$

$$\frac{\partial\phi}{\partial x} = yz^2 ; \frac{\partial\phi}{\partial y} = xz^2 ; \frac{\partial\phi}{\partial z} = 2xyz$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= yz^2\vec{i} + xz^2\vec{j} + 2xyz2\vec{k}$$

$$\nabla\phi)_{(1,0,3)} = 9\vec{j}$$

$$\left|\nabla\phi\right| = \sqrt{81} = 9$$

$\therefore$ The maximum directional derivative of ϕ is $\left|\nabla\phi\right| = 9$

3. Find the magnitude and the direction of the greatest directional derivative of x^2yz^3 at $(2,1,-1)$

Solution :

The direction of the greatest directional derivative is along $\nabla\phi$ and magnitude is $\left|\nabla\phi\right|$

$$\phi = x^2yz^3$$

$$\frac{\partial\phi}{\partial x} = 2xyz^3; \frac{\partial\phi}{\partial y} = x^2z^3; \frac{\partial\phi}{\partial z} = 3x^2yz^2$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$$

$$\nabla\phi)_{(2,1,-1)} = -4\vec{i} + 4\vec{j} + 12\vec{k}$$

$$|\nabla\phi| = \sqrt{16+16+144} = \sqrt{176} = \sqrt{11\times16} = 4\sqrt{11}$$

The direction of the greatest directional derivative is along $\nabla\phi = -4\vec{i} + 4\vec{j} + 12\vec{k}$

4. Find the unit normal to the surface $Z = x^2 + y^2$ at the point (-1,-2,5)

$$\phi = xyz = x^2 + y^2 - z$$

$$\frac{\partial\phi}{\partial x} = 2x; \frac{\partial\phi}{\partial y} = 2y; \frac{\partial\phi}{\partial z} = -1$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\phi)_{(-1,2,5)} = -2\vec{i} - 4\vec{j} - \vec{k}$$

$$|\nabla\phi| = \sqrt{4+16+1} = \sqrt{21}$$

FORMULA unit normal vector to the surface $\phi(xyz) = c$ is $\dfrac{\nabla\phi}{|\nabla\phi|}$

$\therefore$ unit normal vector $\dfrac{= -2\vec{i} - 4\vec{j} - \vec{k}}{\sqrt{21}}$

4. Find the equation of the tangent plane to the surface yz – zx + xy + 5 = 0 at (1,–1,2)

Solution

$$\phi = yz - zx + xy$$

$$\frac{\partial\phi}{\partial x} = -z + y$$

$$\frac{\partial\phi}{\partial y} = z + x$$

$$\frac{\partial\phi}{\partial y} = y - x$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= (y - z)\vec{i} + (x + z)\vec{j} + (y - x)\vec{k}$$

$$\nabla\phi\big)_{(1,-1,2,)} = -3\vec{i} + 3\vec{j} - 2\vec{k}$$

$\therefore$ The direction ratios of the normal to the tangent plane at (1,-1,2) are -3, 3, -2

The tangent plane passes through (1,-1,2) we know that equation of any plane passing through (x_1,y_1,z_1) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0 \qquad \text{------(1)}$$

Here a,b,c = -3,3,-2; (x_1,y_1,z_1) = (1,-1,2)

Using in (1)

Equation of the tangent at (1, -1, 2) is

-3(x-1) + 3(y+1) – d(z-2) = 0.

-3x+3+3y+3-2z+4 = 0

-3x+3y-2z+10 = 0

or 3x – 3y + 2z – 10 = 0

5. Find the angle between the surfaces Z = x^2+y^2-3; and x^2+y^2+z^2 = 9 at the point (2, -1, 2)

Proof:

$\phi_1 = x^2 + y^2 - z$; $\phi_2 = x^2 + y^2 + z^2$;

The angle between ϕ_1 and ϕ_2 is

$$\frac{\nabla\phi_1 . \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|} \qquad \text{---(1)}$$

$\phi_1 = x^2 + y^2 - z$ $\qquad$ $\phi_2 = x^2 + y^2 + z^2$

$\frac{\partial\phi_1}{\partial x} = 2x$ $\qquad$ $\frac{\partial\phi_2}{\partial x} = 2x$

$\frac{\partial\phi_1}{\partial y} = 2y$ $\qquad$ $\frac{\partial\phi_2}{\partial y} = 2y$

$$\frac{\partial \ \phi_{\ 1}}{\partial} = \ - \ 1 \qquad\qquad \frac{\partial\phi_2}{\partial z} = 2z$$

$$\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\phi_1\big)_{(2,-1,2,)} = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} - \overrightarrow{2zk}$$

$$\nabla\phi_2\big)_{(2,-1,2,)} = 4\vec{i} - 2\vec{j} - 4\vec{k}$$

$$|\nabla\phi_1| = \sqrt{16+4+1} = \sqrt{21}$$

$$|\nabla\phi_2| = \sqrt{16+4+16} = \sqrt{36} = 6$$

$$\nabla\phi_1.\nabla\phi_2 = 16$$

Using in (1)

$$\cos\theta = \frac{16}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left[\frac{16}{3\sqrt{21}}\right]$$

6. Find the function ϕ if

$$\nabla\phi = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 - 3x^2yz^2)\vec{k}$$

Solution

By definition

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\therefore \quad \frac{\partial\phi}{\partial x} = y^2 - 2xyz^3; \qquad \text{---(1)}$$

$$\frac{\partial\phi}{\partial y} = 3 + 2xy - x^2z^3 \qquad \text{---(2)}$$

$$\frac{\partial\phi}{\partial z} = 6z^3 + 3x^2yz^2$$

Integrating both side of (1), (2), (3) w.r.t. x,y,z resply.

$\phi = xy^2 - x^2yz^3 +$ a function not containing x

$\phi = 3y + xy^2 - x^2yz^3 +$ a function not containing y

$\phi = \frac{3}{2}z^4 - x^2yz^3 +$ a function not containing z

The general form of ϕ is

$$\phi = 3y + \frac{3}{2}z^4 + xy^2 - x^2yz^3 + c$$

7. Find $\nabla.\vec{r}$ if $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$$f_1 = x; \quad f_2 = y; \quad f_3 = z$$

$$\frac{\partial f_1}{\partial x} = 1 \qquad \frac{\partial f_2}{\partial y} = 1 \qquad \frac{\partial f_3}{\partial z} = 1$$

$$\nabla.\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

= 1+1+1 = 3.

9. Find $\nabla.\vec{r}$ if $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$= f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$$f_1 = x; \quad f_2 = y; \quad f_3 = z$$

$$\frac{\partial f_1}{\partial y} = 0 \qquad \frac{\partial f_2}{\partial z} = 0 \qquad \frac{\partial f_3}{\partial x} = 0$$

$$\frac{\partial f_1}{\partial z} = 0 \qquad \frac{\partial f_2}{\partial x} = 0 \qquad \frac{\partial f_3}{\partial y} = 0$$

$$\nabla \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right]\vec{i} - \left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right]\vec{j} + \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]\vec{k}$$

$$= \vec{\partial}$$

9. If $\vec{f} = x^2 y\vec{i} - 2xz\vec{j} + 2yz\vec{k}$, find (i) $\nabla.\vec{f}$ (ii) $\nabla \times \vec{f}$

Solution (i) $\vec{f} = x^2 y\vec{i} - 2xz\vec{j} + 2yz\vec{k}$

$$= f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$$f_1 = x^2 y; \quad f_2 = -2xz; \quad f_3 = 2yz$$

$$\frac{\partial f_1}{\partial y} = 2xy \qquad \frac{\partial f_2}{\partial z} = 0 \qquad \frac{\partial f_3}{\partial x} = 2y$$

$$\nabla.\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= 2xy + 0 + 2y$$

$$= 2xy + 2y$$

$$= 2y(x+1)$$

(ii) $f_1 = x^2 y; \quad f_2 = -2xz; \quad f_3 = 2yz$

$$\frac{\partial f_1}{\partial y} = x^2 \qquad \frac{\partial f_2}{\partial z} = -2x \qquad \frac{\partial f_3}{\partial x} = 0$$

$$\frac{\partial f_1}{\partial z} = 0 \qquad \frac{\partial f_2}{\partial x} = -2z \qquad \frac{\partial f_3}{\partial y} = 2z$$

$$\nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right]\vec{i} - \left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right]\vec{j} + \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]\vec{k}$$

$$= [2z + 2x]\vec{i} - [0 - 0]\vec{j} + [-2z - x^2]\vec{k}$$

$$= [2x + 2z]\vec{i} - [x^2 + 2z]\vec{k}$$

10. Find the constants a,b,c so that the vector

$\vec{f} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ is irrotational.

Solution: $\vec{f} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$

$$= f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$$f_1 = x + 2y + az; \quad f_2 = bx - 3y - z; \quad f_3 = 4x + cy + 2z$$

$$\frac{\partial f_1}{\partial y} = z \qquad \frac{\partial f_2}{\partial z} = -1 \qquad \frac{\partial f_3}{\partial x} = 4$$

$$\frac{\partial f_1}{\partial z} = a \qquad \frac{\partial f_2}{\partial x} = b \qquad \frac{\partial f_3}{\partial y} = c$$

$$\nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right]\vec{i} - \left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right]\vec{j} + \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]\vec{k}$$

$$= [c + 1]\vec{i} - [4 - a]\vec{j} + [b - 2]\vec{k}$$

$\vec{f}$ is irrotational $\nabla \times \vec{f} = \vec{0}$

$\therefore \quad [c + 1]\vec{i} - [4 - a]\vec{j} + [b - 2]\vec{k} = \vec{0}$

$\therefore \quad c + 1 = 0; \qquad 4 - a = 0; \qquad b - 2 = 0$

$c = -1; \qquad a = 4; \qquad b = 2$

Hence $a = 4, b = 2, c = -1;$

11. Prove that $r^n\vec{r}$ is irrotational for any value of n.

Proof : $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$r = \left|\vec{r}\right| = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\therefore \quad 2r\frac{\partial r}{\partial x} = 2x \qquad 2r\frac{\partial r}{\partial y} = 2y \qquad 2r\frac{\partial r}{\partial z} = 2z$$

$$\frac{\partial r}{\partial x} = {}^{x}\!/_{r} \qquad \frac{\partial r}{\partial y} = {}^{y}\!/_{r} \qquad \frac{\partial r}{\partial z} = {}^{z}\!/_{r}$$

$$\vec{f} = r^n \vec{r}$$

$$= r^n (x\vec{i} + y\vec{j} + z\vec{k})$$

$$= r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}$$

$$= f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$$f_1 = r^n x$$

$$\frac{\partial f_1}{\partial y} = xnr^{n-1}\frac{\partial r}{\partial y} = x.nr^{n-1}.\frac{y}{r}$$

$$= nxyr^{n-2}$$

$$\frac{\partial f_1}{\partial z} = xnr^{n-1}\frac{\partial z}{\partial r} = nxzr^{n-2}$$

$$f_2 = r^n y$$

$$\frac{\partial f_2}{\partial z} = ynr^{n-1}\frac{\partial r}{\partial z} = nyzr^{n-2}$$

$$\frac{\partial f_2}{\partial x} = ynr^{n-1}\frac{\partial r}{\partial x} = yxnr^{n-2}$$

$$f_2 = r^n z$$

$$\frac{\partial f_3}{\partial x} = zxnr^{n-1}\frac{\partial r}{\partial n} = nxzr^{n-1}$$

$$\frac{\partial f_3}{\partial y} = znr^{n-1}\frac{\partial r}{\partial y} = yznr^{n-2}$$

$$\nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right]\vec{i} - \left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right]\vec{j} + \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]\vec{k}$$

$$= \left[nyzr^{n-2} - nyzr^{n-2}\right]\vec{i} - \left[nxzr^{n-2} - nyzr^{n-2}\right]\vec{j} + \left[xynr^{n-2} - xynr^{n-2}\right]\vec{k}$$

$$= \vec{0}$$

$\therefore$ $\vec{f}$ is irrotational for any value of

6.2.1. Important vector identities

1. Prove that, if g be a differentiable scalar point function and $\vec{f}$ be a differentiable vector point function $\nabla.(\vec{f}g) = (\nabla g).\vec{f} + g(\nabla.\vec{f})$

Proof:

$$\nabla.(\vec{f}g) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(\vec{f}g\right)$$

$$= \vec{i}.\frac{\partial}{\partial x}\left(\vec{f}g\right) + \vec{j}\frac{\partial}{\partial y}.\left(\vec{f}g\right) + \vec{k}\frac{\partial}{\partial z}\left(\vec{f}g\right)$$

$$= \vec{i}.\left[\frac{\partial \vec{f}}{\partial x}g + \vec{f}\frac{\partial g}{\partial x}\right] + \vec{j}.\left[\frac{\partial \vec{f}}{\partial y}g + \vec{f}\frac{\partial g}{\partial y}\right] + \vec{k}.\left[\frac{\partial \vec{f}}{\partial z}g + \vec{f}\frac{\partial g}{\partial z}\right]$$

$$= \left[\vec{i}.\frac{\partial \vec{f}}{\partial x} + \vec{j}.\frac{\partial \vec{f}}{\partial y} + \vec{k}.\frac{\partial \vec{f}}{\partial z}\right]\vec{g} + \left[\vec{i}\frac{\partial g}{\partial x} + \vec{j}\frac{\partial g}{\partial y} + \vec{k}\frac{\partial g}{\partial z}\right].\vec{f}$$

$$= \vec{i}.\left[\frac{\partial \vec{f}}{\partial x} \times \vec{g} - \frac{\partial \vec{g}}{\partial x} \times \vec{f}\right] + \vec{j}.\left[\frac{\partial \vec{f}}{\partial y} \times \vec{g} - \frac{\partial \vec{g}}{\partial y} \times \vec{f}\right] + \vec{k}.\left[\frac{\partial \vec{f}}{\partial z} \times \vec{g} - \frac{\partial \vec{g}}{\partial z} \times \vec{f}\right]$$

$$= \vec{i}.\left(\frac{\partial \vec{f}}{\partial x} \times \vec{g}\right) + \vec{j}.\left(\frac{\partial \vec{f}}{\partial y} \times \vec{g}\right) + \vec{k}.\left(\frac{\partial \vec{f}}{\partial z} \times \vec{g}\right)$$

$$-\left[\vec{i}.\left(\frac{\partial \vec{g}}{\partial x} \times \vec{f}\right) + \vec{j}\left(\frac{\partial \vec{g}}{\partial y} \times \vec{f}\right) + \vec{k}\left(\frac{\partial \vec{g}}{\partial y} \times \vec{f}\right)\right]$$

$$= \left(\vec{i}\times\frac{\partial \vec{f}}{\partial x}\right).\vec{g} + \left(\vec{j}\times\frac{\partial \vec{f}}{\partial y}\right).\vec{g} + \left(\vec{k}\times\frac{\partial \vec{f}}{\partial z}\right).\vec{g} -$$

$$\left[\left(\vec{i}\times\frac{\partial \vec{g}}{\partial x}\right).\vec{f} + \left(\vec{j}\times\frac{\partial \vec{g}}{\partial y}\right).\vec{f} + \left(\vec{k}\times\frac{\partial \vec{g}}{\partial y}\right).\vec{f}\right]$$

$$= \left[\vec{i}\times\frac{\partial \vec{f}}{\partial x} + \vec{j}\times\frac{\partial \vec{f}}{\partial y} + \vec{k}\times\frac{\partial \vec{f}}{\partial z}\right].\vec{g} - \left[\vec{i}\times\frac{\partial \vec{g}}{\partial x} + \vec{j}\times\frac{\partial \vec{g}}{\partial y} + \vec{k}\times\frac{\partial \vec{g}}{\partial y}\right].\vec{f}$$

$$= (\nabla\times\vec{f}).\vec{g} - (\nabla\times\vec{g}).\vec{f}$$

4. Prove that $\nabla.(\vec{f}\times\vec{g}) = (\vec{g}.\nabla).\vec{f} - g(\nabla.\vec{f}) - (\vec{f}.\nabla)\vec{g} + \vec{f}(\nabla.\vec{g})$

Proof:

$$\nabla\times(\vec{f}\times\vec{g}) = \vec{i}\times\frac{\partial}{\partial x}(\vec{f}\times\vec{g}) + \vec{j}\times\frac{\partial}{\partial y}(\vec{f}\times\vec{g}) + \vec{k}\times\frac{\partial}{\partial z}(\vec{f}\times\vec{g})$$

$$= \vec{i}\times\left(\frac{\partial \vec{f}}{\partial x}\times\vec{g} + \vec{f}\times\frac{\partial \vec{g}}{\partial x}\right) + \vec{j}\times\left(\frac{\partial \vec{f}}{\partial y}\times\vec{g} + \vec{f}\times\frac{\partial \vec{g}}{\partial y}\right) + \vec{k}\times\left(\frac{\partial \vec{f}}{\partial z}\times\vec{g} + \vec{f}\times\frac{\partial \vec{g}}{\partial z}\right)$$

$$= \vec{i}\times\left(\frac{\partial \vec{f}}{\partial x}\times\vec{g}\right) + \vec{i}\times\left(\vec{f}\times\frac{\partial \vec{g}}{\partial x}\right) + \vec{j}\times\left(\frac{\partial \vec{f}}{\partial y}\times\vec{g}\right) +$$

$$\vec{j}\times\left(\vec{f}\times\frac{\partial \vec{g}}{\partial y}\right) + \vec{k}\times\left(\frac{\partial \vec{f}}{\partial z}\times\vec{g}\right) + \vec{k}\times\left(\vec{f}\times\frac{\partial \vec{g}}{\partial z}\right)$$

$$= (\vec{i}.\vec{g})\frac{\partial \vec{f}}{\partial x} - \left(\vec{i}.\frac{\partial \vec{f}}{\partial x}\right)\vec{g} + \left(\vec{i}.\frac{\partial \vec{g}}{\partial x}\right)\vec{f} - (\vec{i}.\vec{f})\frac{\partial \vec{f}}{\partial x} + (\vec{j}.\vec{g})\frac{\partial \vec{f}}{\partial y} - \left(\vec{j}.\frac{\partial \vec{f}}{\partial y}\right)\vec{g}$$

$$+ \left(\vec{j}.\frac{\partial \vec{g}}{\partial y}\right)\vec{f} - (\vec{j}.\vec{f})\frac{\partial \vec{g}}{\partial y} + (\vec{k}.\vec{g})\frac{\partial \vec{f}}{\partial z} - \left(\vec{k}.\frac{\partial \vec{f}}{\partial z}\right)\vec{g} + \left(\vec{k}.\frac{\partial \vec{g}}{\partial z}\right)\vec{f} - (\vec{k}.\vec{f})\frac{\partial \vec{g}}{\partial z}$$

$$= (\vec{g}.\vec{i})\frac{\partial \vec{f}}{\partial x} + (\vec{g}.\vec{j})\frac{\partial \vec{f}}{\partial y} + (\vec{g}.\vec{k})\frac{\partial \vec{f}}{\partial z} - \left[\left(\vec{i}.\frac{\partial \vec{f}}{\partial x}\right)\vec{g} + \left(\vec{j}.\frac{\partial \vec{f}}{\partial y}\right)\vec{g} + \left(\vec{k}.\frac{\partial \vec{f}}{\partial z}\right)\vec{g}\right]$$

$$- \left[(\vec{f}.\vec{i})\frac{\partial \vec{g}}{\partial x} + (\vec{f}.\vec{j})\frac{\partial \vec{g}}{\partial y} + (\vec{f}.\vec{k})\frac{\partial \vec{g}}{\partial z}\right] + \left(\vec{i}.\frac{\partial \vec{g}}{\partial x}\right)\vec{f} + \left(\vec{j}.\frac{\partial \vec{g}}{\partial y}\right)\vec{f} + \left(\vec{k}.\frac{\partial \vec{g}}{\partial z}\right)\vec{f}$$

$$= \left(\vec{g}.\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\vec{f} - \left[\vec{i}.\frac{\partial \vec{f}}{\partial x} + \vec{j}.\frac{\partial \vec{f}}{\partial y} + \vec{k}.\frac{\partial \vec{f}}{\partial z}\right]\vec{g} -$$

$$\left(\vec{f}.\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\vec{g} - \left[\vec{i}.\frac{\partial \vec{g}}{\partial x} + \vec{j}.\frac{\partial \vec{g}}{\partial y} + \vec{k}.\frac{\partial \vec{g}}{\partial z}\right]\vec{f}$$

$$= \left(\vec{g}.\nabla\right)\vec{f} - \left[\nabla.\vec{f}\right]\vec{g} - \left(\vec{f}.\nabla\right)\vec{g} + \left[\nabla.\vec{g}\right]\vec{f}$$

5. $\nabla.(\vec{f}.\vec{g}) = (\vec{g}.\nabla).\vec{f} + (\vec{f}.\nabla)\vec{g} + \vec{g}\times(\nabla\times\vec{f}.) + \vec{f}\times(\nabla\times\vec{g})$

Proof:

$$\nabla(\vec{f}.\vec{g}) = \vec{i}\frac{\partial}{\partial x}(\vec{f}.\vec{g}) + \vec{j}\frac{\partial}{\partial y}(\vec{f}.\vec{g}) + \vec{k}\frac{\partial}{\partial z}(\vec{f}.\vec{g})$$

$$= \vec{i}.\left(\frac{\partial \vec{f}}{\partial x}.\vec{g} + \vec{f}.\frac{\partial \vec{g}}{\partial x}\right) + \vec{j}\left(\frac{\partial \vec{f}}{\partial y}.\vec{g} + \vec{f}.\frac{\partial \vec{g}}{\partial y}\right) + \vec{k}\left(\frac{\partial \vec{f}}{\partial z}.\vec{g} + \vec{f}.\frac{\partial \vec{g}}{\partial z}\right)$$

$$= \vec{i}\left(\frac{\partial \vec{f}}{\partial x}.\vec{g}\right) + \vec{j}\left(\frac{\partial \vec{f}}{\partial y}.\vec{g}\right) + \vec{k}\left(\frac{\partial \vec{f}}{\partial z}.\vec{g}\right) + \vec{i}\left(\vec{f}.\frac{\partial \vec{g}}{\partial x}\right) + \vec{j}\left(\vec{f}.\frac{\partial \vec{g}}{\partial y}\right) + \vec{k}\left(\vec{f}.\frac{\partial \vec{g}}{\partial z}\right) \quad \text{----(1)}$$

But $\vec{f}\times\left(\frac{\partial \vec{g}}{\partial x}\times\vec{i}\right) = \left(\vec{f}.\vec{i}\right)\frac{\partial \vec{g}}{\partial x} - \left(\vec{f}.\frac{\partial \vec{g}}{\partial x}\right)\vec{i}$

$$\therefore \quad \left(\vec{f}.\frac{\partial \vec{g}}{\partial x}\right)\vec{i} = \left(\vec{f}.\vec{i}\right)\frac{\partial \vec{g}}{\partial x} - \vec{f}\times\left(\frac{\partial \vec{g}}{\partial x}\times\vec{i}\right)$$

$$= \left(\vec{f}.\vec{i}\right)\frac{\partial \vec{g}}{\partial x} + \vec{f}\times\left(\vec{i}\times\frac{\partial \vec{g}}{\partial x}\right)$$

$$\sum\left(\vec{f}.\frac{\partial \vec{g}}{\partial x}\right)\vec{i} = \sum\left(\vec{f}.\vec{i}\right)\frac{\partial \vec{g}}{\partial x} + \sum\vec{f}\times\left(\vec{i}\times\frac{\partial \vec{g}}{\partial x}\right)$$

Also $\vec{g} \times \left(\vec{i} \times \frac{\partial \vec{f}}{\partial x} \right) = \left(\vec{g} . \frac{\partial \vec{f}}{\partial x} \right) \vec{i} - \left(\vec{g} . \vec{i} \right) \frac{\partial \vec{f}}{\partial x}$

$$\therefore \quad \left(\vec{g} . \frac{\partial \vec{f}}{\partial x} \right) \vec{i} = \vec{g} \times \left(\vec{i} . \frac{\partial \vec{f}}{\partial x} \right) + \left(\vec{g} . \vec{i} \right) \frac{\partial \vec{f}}{\partial x}$$

$$\therefore \quad \sum \left(\vec{g} . \frac{\partial \vec{f}}{\partial x} \right) \vec{i} = \sum \vec{g} \times \left(\vec{i} . \frac{\partial \vec{f}}{\partial x} \right) + \sum \left(\vec{g} . \vec{i} \right) \frac{\partial \vec{f}}{\partial x}$$

Using in (1)

$$\nabla . (\vec{f} . \vec{g}) = \sum \vec{g} \times \left(\vec{i} \times \frac{\partial \vec{f}}{\partial x} \right) + \sum \left(\vec{g} . \vec{i} \right) \frac{\partial \vec{f}}{\partial x} + \sum \left(\vec{f} . \vec{i} \right) \frac{\partial \vec{g}}{\partial x} + \sum \vec{f} \times \left(\vec{i} \times \frac{\partial \vec{g}}{\partial x} \right)$$

$$= \vec{g} \times (\nabla \times \vec{f}) + (\vec{g} . \nabla) \vec{f} + (\vec{f} . \nabla) \vec{g} + \vec{f} \times (\nabla \times \vec{g})$$

6.2.2 Second order differential operation

The operator ∇^2 is defined as

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

∇^2 is called Laplacian operator

$$\nabla . (\nabla \phi) = \nabla . \left(\vec{i} \frac{\partial \varphi}{\partial x} + \vec{j} \frac{\partial \varphi}{\partial y} + \vec{k} \frac{\partial \varphi}{\partial z} \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \varphi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial z} \right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Example 1 : Prove that $\nabla . (\nabla \phi) = \vec{0}$ (or) Curl (grad ϕ) $= \vec{0}$

Proof:

$$\nabla \phi = \vec{i} \frac{\partial \varphi}{\partial x} + \vec{j} \frac{\partial \varphi}{\partial y} + \vec{k} \frac{\partial \varphi}{\partial z}$$

$$\nabla\times\nabla\phi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix}$$

$$= \left[\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial y\partial z}\right]\vec{i} - \left[\frac{\partial^2\varphi}{\partial x\partial z} - \frac{\partial^2\varphi}{\partial z\partial x}\right]\vec{j} + \vec{k}\left[\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y\partial x}\right]$$

$$= \vec{0}$$

$$\nabla\times\nabla\phi) = \vec{0}$$

2. Prove that $\nabla.(\nabla\times\vec{f}) = 0$ (or) dir curl $\vec{f} = 0$

Proof:

$$\vec{f} = f_1\vec{i} + f_2\vec{j}\frac{\partial\varphi}{\partial y} + f_3\vec{k}\frac{\partial\varphi}{\partial z}$$

$$\nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right]\vec{i} - \left[\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right]\vec{j} + \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]\vec{k}$$

$$= \left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right]\vec{i} + \left[\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right]\vec{j} + \left[\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right]\vec{k}$$

$$\therefore \nabla.(\nabla\times\vec{f}) = \frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= \frac{\partial^2 f_3}{\partial x\partial y} - \frac{\partial^2 f_2}{\partial x\partial z} + \frac{\partial^2 f_1}{\partial y\partial z} - \frac{\partial^2 f_3}{\partial y\partial x} + \frac{\partial^2 f_2}{\partial z\partial x} - \frac{\partial^2 f_1}{\partial z\partial y}$$

$$= 0$$

$$\nabla.(\nabla.\vec{f}) = 0$$

3. Prove that $\nabla^2 r^n = n(n+1)r^{n-2}$

Proof: $f = r^n$ $\qquad |\vec{r}| = \sqrt{x^2+y^2+z^2} \qquad \therefore r^2 = x^2+y^2+z^2$

$$\nabla^2 . f = \nabla.(\nabla f)$$

$$= \nabla.\left[\vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}\right] \qquad \text{---(1)}$$

$$f = r^n$$

$$\frac{\partial f}{\partial x} = nr^{n-1}\frac{\partial r}{\partial x} = nr^{n-1}\frac{x}{r}$$

$$= nxr^{n-2}$$

$$\frac{\partial f}{\partial y} = nr^{n-1}\frac{\partial r}{\partial y} = nr^{n-1}\frac{y}{r}$$

$$= nyr^{n-2}$$

$$\frac{\partial f}{\partial z} = nr^{n-1}\frac{\partial r}{\partial z} = nr^{n-1}\frac{z}{r}$$

$$= nzr^{n-2}$$

$$\nabla.(\nabla f) = \nabla.\left[nxr^{n-2}\vec{i} + nyr^{n-2}\vec{j} + nzr^{n-2}\vec{k}\right]$$

$$\frac{\partial}{\partial x}\left(nxr^{n-2}\right) + \frac{\partial}{\partial y}\left(nyr^{n-2}\right) + \frac{\partial}{\partial z}\left(nzr^{n-2}\right)$$

$$= n\left[x(n-2)r^{n-3}\frac{\partial r}{\partial n} + r^{n-2}.1 + y(n-2)r^{n-3}.\frac{y}{r} + r^{n-2} + z(n-2)r^{n-3}\frac{\partial r}{\partial z} + r^{n-2}.1\right]$$

$$= n\left[3r^{n-2} + x(n-2)r^{n-3}\frac{r}{r} + y(n-2)r^{n-3}.\frac{y}{r} + z(n-2)r^{n-3}\frac{z}{r}\right]$$

$$= n\left[3r^{n-2} + x^2(n-2)r^{n-4} + y^2(n-2)r^{n-4} + z^2(n-2)r^{n-4}\right]$$

$$= n\left[3r^{n-2} + (n-2)r^{n-4}(x^2+y^2+z^2)\right]$$

$$= n\left[3r^{n-2} + (n-2)r^{n-4}r^2\right]$$

$$= n\left[3r^{n-2} + (n-2)r^{n-2}\right]$$

$$= n\left[3+n-2\right]r^{n-2}$$

$$\nabla.\nabla r^n = n(n+1)r^{n-2}$$

$$\therefore \nabla^2(r^n) = n(n+1)r^{n-2}$$

6.3 Let us sum up

So far we have studied the concept of $\bar{y}.\vec{\ }\bar{f},\vec{\ }\bar{y}\vec{\ }$ and $\bar{y}\vec{\ }$. Also the definition of solenoidal vectors and irrotational vectors.

6.4 Check your progress

(1) Find f = $x^2y^2z^2$

(2) Find $\bar{f}\,a\,\vec{r}\,e\,\bar{y}\,x\,\vec{f}\,i\,f$
$f = x\,\vec{i} + y\,\vec{i} + z\,\vec{\ }$

6.5. Lesson End Activities

1. Find $\nabla\phi$ if $\phi = 3x^2y - y^3z^2$ at (1,-2,-1)
2. Find a unit normal to the surface $x^2y + 2xz = 4$ at (2,-2,3)
3. If $\vec{a}$ is a constant vector and $\vec{r}$ is the position vector of any point (x,y,z), prove that (i) $\nabla(\vec{a}.\vec{r}) = \vec{a}$ (ii) $\nabla.(\vec{a}\times\vec{r}) = 0$ (iii) $(\vec{a}.\nabla)\vec{r} = \vec{a}$ (iv) $\nabla\times(\vec{a}\times\vec{r}) = 2\vec{a}$
4. Find the directional derivative of $4xz^2 + x^2yz$ at (1,-2,-1) in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.
5. Find the maximum directional derivative of $x^2 + y^2 + z^2$ at (2,-2,-2)
6. Find the magnitude and the direction of the greatest directional derivative of $xy + yz + zx$ at (1,1,3)
7. Find the unit vector normal to the surface $x^4 - 3xyz + z^2 + 1 = 0$ at (1,1,1)
8. Find the equation of the tangent plane to the surface $x^2 + y^2 + z^2 = 25$ at the point (4,0,3)
9. Find the angle of intersection of the surfaces $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 = 0$ at (4,-3,2)
10. Find $\nabla.\vec{f}$ and $\nabla\times\vec{f}$ if $\vec{f} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ at (1,-1, 1)
11. If $\vec{f} = (x + y + 1)\vec{i} + \vec{j} - (-x - y)\vec{k}$, prove that $\vec{f}.(\nabla\times\vec{f}) = 0$

12. Determine the constant a so that the vector

$\vec{f} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}$, is solenoidal.

13. Show that the vector $\vec{f} = (\sin y + z)\vec{i} + (x\cos y - z)\vec{j} + (x-y)\vec{k}$ is irrotational.

14. Find the constants a,b,c so that the vector

$\vec{f} = (axy - z^3)\vec{i} + (a-z)x^2\vec{j} + (1-a)xz^2\vec{k}$ is irrotational

15. Show that $\vec{f} = (2x^2 + 8xy^2 z)\vec{i} + (3x^3 y - 3xy)\vec{j} + (4y^2 z^2 + 2x^3 z)\vec{k}$, is not solenoidal

but $g = xyz^2\vec{f}$ is solenoidal.

6.6 Points for discussion

1. Let $\vec{a}$ be a constant vector. Prove that

(i) $\nabla(\vec{a}\times\vec{r}) = 2\vec{a}$ (ii) $\nabla.(r^3\vec{r}) = 6r^3$ (iii) $\nabla\times\left(\frac{\vec{r}}{r^2}\right) = \frac{2}{r^2}\vec{r}$ (iv) $\nabla\left[r\nabla\left(\frac{1}{r^3}\right)\right] = \frac{3}{r^4}$

2. Prove that $\nabla r^n = nr^{n-2}\vec{r}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$

3. Prove that $\nabla.(r^n\vec{r}) = (n+3)r^n$

4. For what value of n, $\nabla.(r^n\vec{r}) = 0$

5. Prove that $\nabla\times\left\lfloor f(r)\vec{r}\right\rfloor = \vec{0}$ if f(r) its differentiable

6.7 Sources

1) Vector Calculus by - Namasivayam
2) Vector Calculus by - Rasinghmia aggarval
3) Vector calculus by - P. Durai Pandian
4) Vector calculus by - Chatterjee

Lesson 7

Contents

7.0Aim and Objectives

In this lesson we are going to study about the different types of integrals in vector calculus (viz) line integral, surface integral and volume integral using the fundamentals of integration.

7.1. Integration of Vector's

Line Integral

Let $\vec{f}(x, y, z)$ be a vector point function defined through out some region of space then $\int_c \vec{f}.\vec{dr}$ is defined as the line integral of $\vec{f}$ along C. Where C is any curve in that region.

Also $\int_A^B \vec{f}.d\vec{r}$ is called the tangential line integral over C from A to B.

$$\text{Let } \vec{f} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{dr} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\therefore \quad \vec{f}.\vec{dr} = f_1dx + f_2dy + f_3dz$$

$$\therefore \qquad \int_C \vec{f}.\overrightarrow{dr} = \int_C \left(f_1 dx + f_2 dy + f_3 dz \right)$$

More over $\int_C \vec{f} \times \overrightarrow{dr}$ and $\int_C \phi \overrightarrow{dr}$ are also line integrals.

Problems:

1. Evaluate $\int_C \vec{f}.\overrightarrow{dr}$ where $\vec{f} = x^2 \vec{i} + y^3 \vec{j}$ *along* $y = x^2$ from (0,0) to (1,1)

Solution : The limits for x are x = 0 to x = 1

$y = x^2$

$dz = 2xdx$ ---(1)

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$\overrightarrow{dr} = dx\vec{i} + dy\vec{j}$$

$$\vec{f}.\overrightarrow{dr} = \left(x^2\vec{i} + y^3\vec{j}\right).(dx\vec{i} + dy\vec{j})$$

$$= x^2 dx + y^3 dy$$

$$= x^2 dx + x^6 .2xdx, \text{ using (1)}$$

$$= x^2 dx + 2x^7 dx,$$

$$\therefore \int_C \vec{f}.\overrightarrow{dr} = \int_0^1 x^2 dx + 2x^7 dx$$

$$= \left.\frac{x^3}{3}\right]_0^1 + 2\left(x\frac{8}{8} \right)_0^1$$

$$= \frac{1}{3} + \frac{2}{8} = \frac{14}{24} = \frac{7}{12}$$

2. If $\vec{f} = (3x^2 + 6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$, evaluate $\int_C \vec{f}.\overrightarrow{dr}$ where C is a straight line joining (0,0,0) to (1,1,1)

Solution : Step: Equation of the line joining the points: (x_1,y_1,z_1) and (x_2,y_2,z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - x_1}{y_2 - y_1} = \frac{z - x_1}{x_2 - x_1}$$

$$\frac{x-0}{1}=\frac{y-0_1}{1}=\frac{z-0}{1}$$

$x=y=z=t(say)$

$\therefore \quad x=t, y=t, z=t$

$At(0,0,0), t=0, at(1,1,1)t=1$

$\therefore \quad t=0 \; to \; t=1$

$$\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$$

$$\vec{r}=t\left(\vec{i}+\vec{j}+\vec{k}\right)$$

$$\overrightarrow{dr}=\left(\vec{i}+\vec{j}+\vec{k}\right)dt$$

$$\vec{f}=\left(3x^2+6y\right)\vec{i}+14yz\vec{j}+20xz^2\vec{k}$$

$$\therefore \quad \vec{f}=\left(3t^2+6t\right)\vec{i}+14t^2\vec{j}+20t^3\vec{k}$$

$$\vec{f}.\overrightarrow{dr}=3t^2+6t-14t^2+20t^3$$

$$=20t^3-11t^2+6t$$

$$\therefore \int_C \vec{f}.\overrightarrow{dr}=\int_0^1\left(20t^3-11t^2+6t\right)dt$$

$$=\frac{13}{3}$$

7.2. Surface Integral

In order to evaluate surface integrals it is convenient to express them as double integrals taken over the orthogonal projection of the surface S on the line of coordinate places. Let S be a given surface. Let $\vec{f}$ be the vector point function. Let $\vec{n}$ be the unit outward drawn normal vector to the surface S. Then the surface integral is defined as

$$\iint_S \vec{f}.\vec{n}ds$$

a. If the position is on the xy plane, then the surface integral is $\iint_R \vec{f}.\vec{n}\frac{dxdy}{\left|\vec{n}.\vec{x}\right|}$

where R is the orthogonal projection.

b. If the Orthogonal projection R is on yz plane then the surface integral is

$$\int\int_R \vec{f}.\vec{n}\,\frac{dydz}{\left|\vec{n}.\vec{j}\right|}$$

c. If the Orthogonal projection R is on zx plane then the surface integral is $\int\int_R \vec{f}.\vec{n}\frac{dzdx}{\left|\vec{n}.\vec{k}\right|}$

Examples:

1. Evaluate $\int\int_S \vec{f}.\vec{n}ds$ where $\vec{f} = z\vec{i} + x\vec{j} + 3y^2z\vec{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between z =0 and z = 5.

Solution :

$$\phi = x^2 + y^2; \qquad \frac{\partial\varphi}{\partial x} = 2x; \qquad \frac{\partial\varphi}{\partial y} = 2y$$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y}$$

$$= 2x\vec{i} + 2y\vec{j}$$

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4x^2 + 4y^2}}$$

$$= \frac{2\left(x\vec{i} + y\vec{j}\right)}{2\sqrt{x^2 + y^2}}$$

$$= \frac{x\vec{i} + y\vec{j}}{\sqrt{16}}$$

$$\vec{n} = \frac{1}{4}\left(x\vec{i} + y\vec{j}\right)$$

Let R be the projection of S on xz plane

$$\int\int_S \vec{f}.\vec{n}ds = \int\int_R \vec{f}.\vec{n}\frac{dxdz}{\left|\vec{n}.\vec{j}\right|} \qquad \text{---(1)}$$

$$\vec{f}.\vec{n} = \left(z\vec{i} + x\vec{j} - 3y^2z\vec{k}\right).\frac{1}{4}(x\vec{i} + y\vec{j})$$

$$= \frac{1}{4}(xz + xy)$$

$$= \frac{1}{4}x(y + z)$$

$$\vec{n}.\vec{j} = \frac{1}{4}(x\vec{i} + y\vec{j}).\vec{j} = \frac{y}{4}$$

$\therefore$ using in (1) $\iint_S \vec{f}.\vec{n}ds = \iint_R \frac{\frac{1}{4}x(y+z)}{y/4} 3$

$$\int_{z=0}^{5}\int_{z=0}^{4} \frac{xz + xy}{y}$$

$$= \int_{z=0}^{5}\int_{z=0}^{4} \left(\frac{xz}{y} + x \right) dxdz$$

$$= \int_{z=0}^{5}\int_{z=0}^{4} \left(\frac{xz}{\sqrt{16 - x^2}} + x \right) dxdz$$

$$= 90$$

7.3. Volume Integral

Let V be a volume bounded by a surface suppose f(x,y,z) is a single valued function of position defined over V.

Then $\iiint_V f(x, y, z)dv$ is defined as a volume integral.

If we sub divide the volume V into small cuboids by drawing lines parallel to the three co-ordinate axes, then $dv = dxdydz$ and the volume integral becomes

$$\iiint_V f(x, y, z)dxdydz$$

If $\vec{f}$ is a vector point function, then $\iiint_V \vec{f} dv$

Example: Evaluate $\iiint_V \phi dv$ where $\phi = 45x^2y$ and v is the closed region bounded by the planes

$4x + 2y + 2 = 8;\ x = 0, y = 0, z = 0$

Solution: We have

$$\iiint_V \phi dv = \int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} 45x^2 ydxdydz$$

In the equation $4x+2y+2=8$,

Put y =0, z = 0, 4x = 8

x = 2

$\therefore \quad x = 0 \; to \; x = 2$

Put z = 0,

$4x+2y+2=8$

$2x+y=8$

$y=8-2x$

$\therefore \quad y = 0 \; to \; y = 8-2x$

Also from

$4x+2y+2=8$

$z=8-4x-2y$

$\therefore \quad z = 0 \; to \; y = 8-4x-2y$

$$\iiint_V \phi dv = 45 \int_{x=0}^{2} \int_{y=0}^{4-2x} x^2 y \left(z\right)_0^{8-4x-2y} dxdy$$

$$= 45 \int_0^2 \int_0^{4-2x} x^2 y(8x-4x-2y)dydx$$

$$= 45 \int_0^2 \int_0^{4-2x} (8x^2 y - 4x^3 y - 2x^2 y^2)dydx$$

$$= 45 \int_0^2 \int_0^{4-2x} \left(8x^2 \frac{y^2}{2} - 4x^3 \frac{y^2}{3} - 2x^2 \frac{y^3}{3} \right)_0^{4-2x} dx$$

$$= 45 \int_0^2 \left[4x^2 (4-2x)^2 - 2x^3 (4-2x)^2 - 2\frac{x^3}{3}(4-2x)^3 \right] dx$$

$$= 128$$

7.4 Let us sum up

We have studied so far evaluating the line integral surface integrals and volume integrals, with the fundamentals of integration of vectors.

7.5 Check your progress

1. Evaluate $\int_C \vec{f}.\vec{dr}$ where $\vec{f} = x^2\vec{i} + y^3\vec{j}$ and the curve C is the arc of the parabola y = x^2 in the xy plane from (0,0) to (1,1)

$$\left(Ans : \frac{7}{12}\right)$$

2. Evaluate $\int_C \vec{f}.\vec{dr}$ where $\vec{f} = (x^2 - y^2)\vec{i} + xy\vec{j}$ and the curve C is the arc of the curve y=x^3 from (0,0) to (2,8)

$$\left(Ans : \frac{821}{21}\right)$$

3. Evaluate $\int (xdy - ydx)$ around the circle $x^2 + y^2 = 1$

$$(Ans : 2\pi)$$

4. Evaluate $\int_C \vec{f}.\vec{dr}$ where $\vec{f} = xy\vec{i} + (x^2 + y^2)\vec{j}$ and the curve C is the arc of the curve y=x^3-4 from (2,0) to (4, 12)

$$(Ans : 732)$$

5. Evaluate $\int\int_S \vec{f}.\vec{n}ds$ where $\vec{f} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ and S is that part of the surface of the sphere $x^2 + y^2 + z_2 = 1$ which lies in the first quadrant

$$\left(Ans : \frac{3}{8}\right)$$

7.6. Points for discussion.

1. Evaluate $\int\int_S \vec{f}.\vec{n}ds$ where $\vec{f} = (x + y^2)\vec{i} + 2x\vec{j} + 2yz\vec{k}$ and S is the surface of the plane $2x + y + 2z = 6$ in the first octant.

$$(Ans : 81)$$

2. Evaluate $\iint_S \vec{f}.\vec{n}ds$ where $\vec{f} = y\vec{i} + 2x\vec{j} - 2\vec{k}$ .Where S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane z = 4.

$(Ans : 108)$

3. If $\vec{f} = (2x^2 - 3)\vec{i} + 2xy\vec{j} - 4x\vec{k}$ then evaluate . $\iiint_V \nabla.\vec{f}dv$ Where V is the closed region bounded by the planes x=0, y=0, z=0, and 2x+2y+z=4.

$\left(Ans : \frac{8}{3}\right)$

7.7 References

1) Vector Calculus by	- Namasivayam
2) Vector Calculus by	- Rasinghmia aggarval
3) Vector calculus by	- P. Durai Pandian
4) Vector calculus by	- Chatterjee

Lesson – 8

Contents

8.0 Aims and Objectives

In this lesson we are going to study about the theorems of gauss, green and stroke’s. These theorems help us in evaluating a particular type of integral.

8.1Theorems of Gauss, Green and Stoke’s.

8.1.1.Green’s Theorem: Let R be a closed bounded region in the xy plane whose boundary C consists of finitely many smooth curves.

Let M and N be continuous function of x and y having continuous partial derivatives $\frac{\partial M}{\partial y}$ *and* $\frac{\partial N}{\partial x}$ in R. Then $\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy = \oint_C (Mdx+Ndy)$

8.1.2. Gauss Theorem: Suppose V is the volume bounded by a closed piece wise smooth surface S. Suppose $\vec{f}=(x,y,z)$ be a vector point function which is continuous and has continuous first order partial derivatives in V. then

$$\iiint_V \nabla.\vec{f}\,dv = \iint_S \vec{f}.\vec{n}\,ds$$

8.1.3.Stoke’s Theorem: Let S be a piece wise smooth open surface bounded by a piece wise smooth simple closed curve C. Let f(x,y,z) be a continuous vector point function which has continuous first order partial derivatives in a region of space which contains S in its interior. Then

$$\oint_C \vec{f}.\vec{dr} = \iint_S (\nabla\times\vec{f}).\vec{n}ds$$

8.2. Examples

1. Gauss Theorem : Verify Gauss Theorem for

$\vec{f} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 4 - xy)\vec{k}$ taken over the rectangular parallelepiped.

$0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c,$

Solution: Step 1. To find $\iiint\limits_V \left(\nabla.\vec{f}\right)dv$

$$\vec{f} = (x^2 - yz)\vec{i} + (y^2 - zx)\vec{j} + (z^2 - xy)\vec{k}$$

$$f_1 = x^2 - yz; \quad f_2 = x^2 - zx; \quad f_3 = z^2 - xy;$$

$$\frac{\partial f_1}{\partial x} = 2x; \qquad \frac{\partial f_2}{\partial y} = 2y; \qquad \frac{\partial f_3}{\partial z} = 2z;$$

$$\nabla.\vec{f} = 2x + 2y + 2z$$

$$= 2\left(x + y + z\right)$$

$$\iiint\limits_V \nabla.\vec{f}\,dv = 2\int_0^a\int_0^b\int_0^c 2(x + y + z)dxdydz$$

$$= 2\int_0^a\int_0^b\int_0^c (x + y + z)dxdydz$$

$$= 2\int_0^a\int_0^b \left[xz + yz + z^2/2\right]_0^c dydz$$

$$= 2\int_0^a\int_0^b \left[cx + ycy + c^2/2\right]dydz$$

$$= 2\int_0^a\int \left[cxy + cy^2/2 + c^2/2\,.y\right]_0^b dx$$

$$= 2\int_0^a\int \left[bcx + b^2\,c/2 + b\,c^2/2\right]dx$$

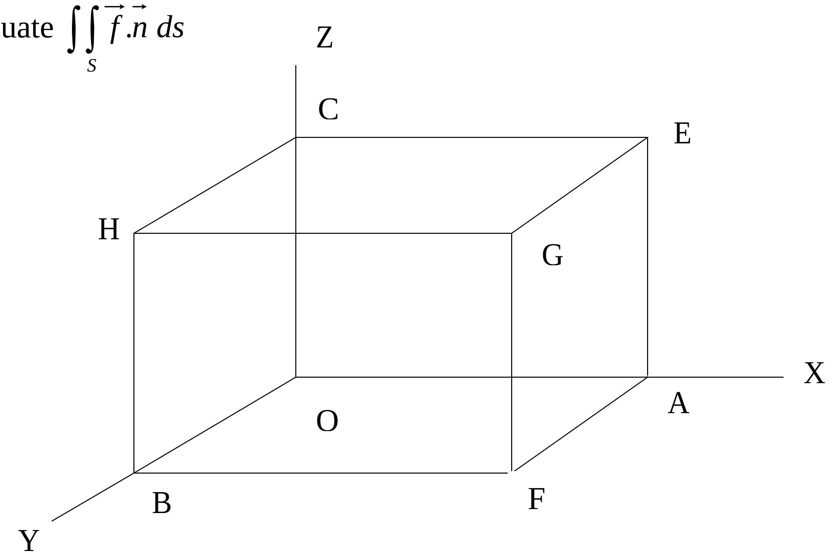
$$= 2\int_0^a \left[\frac{bcx^2}{2} + \frac{b^2cx}{2} + \frac{bc^2x}{2}\right]_0^a$$

$$= 2\left[\frac{a^2bc}{2} + \frac{ab^2c}{2} + \frac{abc^2}{2}\right]$$

$$= 2\frac{abc}{2}[a+b+c]$$

$$= abc[a+b+c] \qquad \text{-------(1)}$$

Step 2: To evaluate $\iint_S \vec{f}.\vec{n}\,ds$



$$\iint_S \vec{f}.\vec{n}\,ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$$

Where S_1 is OAFB, S_2: GECH; S_3: AEGF; S_4: OBHC; S_5: OCEA; S_6: HBFG

To evaluate $\iint_{S_1}$ Put z = 0 in $\vec{f}$

$z = 0 \; in \; \vec{n} = -\vec{k}$ $\qquad \vec{f} = -xy\vec{k}$

$$\therefore \iint_{OAFB} = \int_{y=0}^{b}\int_{x=0}^{a} xydxdz \qquad \vec{f}.\vec{n} = -xy\vec{k}.(-\vec{k})$$

$$= \int_{y=0}^{b}\left.\frac{x^2y}{2}\right]_0^a dy \qquad = xy$$

$$= \frac{1}{2}a^2\int_{y}^{b} ydy$$

$$= \frac{1}{2}a^2\left.\frac{y^2}{2}\right]_0^b = \frac{1}{4}a^2b^2$$

To evaluate S_2

$$\iint_{S_2} = \iint_{GECH} \vec{f}.\vec{n}\, ds \qquad\qquad z = c; \quad \vec{n} = \vec{k}$$

$$\therefore \quad \vec{f}.\vec{n} = (c^2 - xy)$$

$$\iint_{S_2} = \int_{y=0}^{b} \int_{x=0}^{a} (c^2 - xy)dxdz$$

$$= \int_{y=0}^{b} \left[c^2 x - \frac{x^2 y}{2} \right]_0^a dy$$

$$= \int_{y=0}^{b} \left[c^2 a - \frac{a^2 y}{2} \right] dy$$

$$= c^2 ay - \frac{a^2}{2}\frac{y}{2} \Big]_0^b$$

$$= abc^2 - \frac{a^2 b^2}{4}$$

To evaluate $\iint_{S_3} = \iint_{AEGF} \vec{f}.\vec{n}\, ds$

$$x = a; \vec{n} = \pm\vec{i}$$

$$\vec{f}.\vec{n} = +(x^2 - yz)$$

$$= +(a^2 - yz) = -yz + a^2$$

$$\therefore \quad \iint_{S_3} = \int_{y=0}^{b} \int_{z=0}^{c} (-yz + a^2)dzdy$$

$$= \int_{y=0}^{b} \left[-\frac{yz^2}{2} + a^2 z \right]_0^c dz$$

$$= \int_{y=0}^{b} \left[-c^2\frac{y}{2} + a^2 c \right] dy$$

$$= -\frac{c^2 y^2}{4} + a^2 cy \Big]_0^b$$

$$= -\frac{b^2 c^2}{4} + a^2 bc$$

To evaluate $\iint_{S_4} = \iint_{OBHC} \vec{f}.\vec{n}\, ds$

$$x = 0; \vec{n} = -\vec{i}$$

$$\vec{f}.\vec{n} = +(-yz\vec{i} + y^2\vec{j} + 3^2\vec{k}).(-\vec{i})$$

$$= +yz$$

$$\therefore \iint_{S_4} = +\int_{y=0}^{b}\int_{z=0}^{c} yzdzdy$$

$$= +\int_{y=0}^{b}\left[\frac{yz^2}{2}\right]_0^c dz = +\int_{y=0}^{b}\frac{yc^2}{2}dy$$

$$= +\frac{c^2}{2}\frac{y^2}{2}\Big]_0^b$$

$$= +\frac{b^2c^2}{4}$$

To evaluate $\iint_{S_5} = \iint_{OCEA} \vec{f}.\vec{n}\, ds$

$$y = 0; \vec{n} = -\vec{j}$$

$$\vec{f}.\vec{n} = (x^2\vec{i} - zx\vec{j} + z^2\vec{k}).(-\vec{j})$$

$$= +zx$$

$$\therefore \iint_{S_5} = \int_{z=0}^{c}\int_{x=0}^{a} zxdxdz$$

$$= \int_{z=0}^{c}\frac{zx^2}{2}\Big]_0^a dz$$

$$= \frac{1}{2}a^2\int_0^c zdz$$

$$= \frac{1}{2}a^2\frac{z^2}{2}\Big)_0^c$$

$$= \frac{a^2c^2}{4}$$

To Find

$$\iint_{S_6} = \iint_{HBFG} \vec{f}.\vec{n}\, ds \qquad \vec{n} = \vec{j}; \quad y = b$$

$$\therefore \quad \vec{f}.\vec{n} = b^2 - zn$$

$$\therefore \iint_{S_6} = \int_{z=0}^{c} \int_{x=0}^{a} (b^2 - zx)dxdz$$

$$= \int_{z=0}^{c} \left[b^2 x - \frac{zx^2}{2} \right]_0^a dz$$

$$= \int_{z=0}^{c} \left[b^2 a - \frac{a^2 z}{2} \right] dz$$

$$= b^2 az - \frac{a^2}{2}\frac{z^2}{2} \Big]_0^c$$

$$= b^2 ac - \frac{a^2 c^2}{4}$$

$$\therefore \quad \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6} =$$

$$\frac{a^2 b^2}{4} + abc^2 - \frac{a^2 b^2}{4} + \frac{b^2 c^2}{4} + a^2 bc - \frac{b^2 c^2}{4} + \frac{a^2 c^2}{4} + b^2 ac - \frac{a^2 c^2}{4}$$

$$= abc^2 + a^2 bc + ab^2 c$$

$$= abc(a + b + c)$$

$\therefore$ Gauss Theorem is verified.

2. Stoke's theorem:

$$\oint \vec{f}.\vec{dr} = \iint_S (\nabla \times \vec{f}).\vec{n} ds$$

Problem verify stroke's theorem for

$\vec{f} = (2x - y)\vec{i} - yz^2\vec{j} - y^2 z\vec{k}$ where S is the upper half surface of the sphere

$x^2 + y^2 + z^2 = 1$ and C is its boundary

Solution: Step 1: The boundary C of it's a circle in the xy plane of radius = 1 and centre at the origin.

$$x^2 + y^2 = 1$$

$\therefore \quad x = \cos t; \quad y = \sin t; \quad z = 0,$

$\therefore \quad x = \cos t; \quad y = \sin t; \quad z = 0,$

$t = 0 \; to \; 2\pi$

$$\vec{f} = (2x - y)\vec{i} - yz^2\vec{j} - y^2 z\vec{k}$$

$$\vec{f}.\vec{dr} = (2x - y)dx - yz^2 dy - y^2 z dz$$

$x = \cos t \qquad y = \sin t \qquad z = 0$

$dx = -\sin t \; dt \qquad dy = \cos t \; dt \qquad dz = 0$

$$\oint \vec{f}.\vec{dr} = \int_0^{2\pi} (2\cos t - \sin t)(-\sin t)dt$$

$$= \int_0^{2\pi} (-2\sin t \cos t + \sin^2 t)dt$$

$$= \int_0^{2\pi} (2\sin t \cos t \; dt + \int_0^{2\pi} (\sin^2 t \; dt$$

$$= -\int_0^{2\pi} (\sin 2t \; dt + \frac{1}{2}\int_0^{2\pi} (1 - \cos 2t)dt$$

$$= -\left[\frac{-\cos 2t}{2}\right]_0^{2\pi} + \frac{1}{2}\left[\int_0^{2\pi} dt - \int_0^{2\pi} \cos 2t \; dt\right]$$

$$= \frac{1}{2}(1 - 1) + \frac{1}{2}\left(t - \frac{\sin 2t}{2}\right)_0^{2\pi}$$

$$= \frac{1}{2}[(25 - 0) - (0)]$$

$$\oint \vec{f}.\vec{dr} = \pi$$

Step 2: To evaluate $\iint_S (\nabla \times \vec{f}).\vec{n}ds$

$$\vec{f} = (2x - y)\vec{i} - yz^2\vec{j} - y^2 z\vec{k}$$

$$= f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$$

$f_1 = 2x - y \qquad f_2 = -yz^2 \qquad f_3 = -y^2 z$

$$\frac{\partial f_1}{\partial y} = -1 \qquad \frac{\partial f_2}{\partial z} = -2yz \qquad \frac{\partial f_3}{\partial x} = 0$$

$$\frac{\partial f_1}{\partial z} = 0 \qquad \frac{\partial f_2}{\partial x} = 0 \qquad \frac{\partial f_3}{\partial y} = -2yz$$

$$\nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\vec{i} - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_3}{\partial z}\right)\vec{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\vec{k}.$$

$$= (-xyz + xyz)\vec{i} - (0-0)\vec{j} + (0+1)\vec{k}.$$

$$= \vec{k}$$

$$\therefore \quad \nabla \times \vec{f} = \vec{k}$$

Here $\vec{n} = \vec{k}$

$$\therefore \quad \iint_S (\nabla \times \vec{f}).\vec{n}\, ds = \iint_S (\nabla \times \vec{f}).\vec{k}\, ds$$

$$= \iint_S \vec{k}.\vec{k}\, ds$$

$$= \int_S \int_S ds = \pi$$

$\therefore$ Stoke’s theorem is verified.

Green’s Theorem:

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_C Mdx + Ndy$$

Verify Green’s theorem in the plane for $\oint_C (xy + y^2)dx + x^2 dy$ where C is the closed curve of the region bounded by y = x and y = x^2

Solution : Step 1

$$M = xy = y^2 \qquad\qquad N = x^2$$

$$\frac{\partial M}{\partial y} = 2y + x \qquad\qquad \frac{\partial N}{\partial x} = 2x$$

$y = x$ ---(1) and $y = x^2$ ---(2)

(1) and (2) intersect at (0,0) and (1,1)

$$\therefore \qquad x = 0 \; to \; x = 1$$

$$y = x^2 \; to \; x$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \int_{x=0}^{1} \int_{y=x^2}^{x} [2x - x - 2y] dydx$$

$$= \int_{x=0}^{1} \int_{y=x^2}^{x} [x - 2y] dydx$$

$$= \int_{x=0}^{1} \left[xy - y^2 \right]_{x^2}^{x} dx$$

$$= \int_{0}^{1} \left[\left(x^2 - x^2 \right) - (x^3 - x^4) \right] dx$$

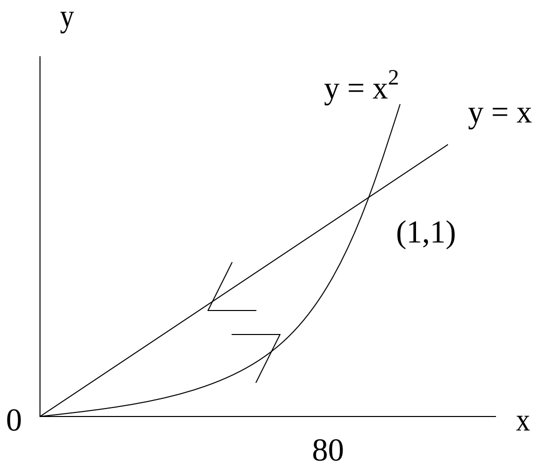
$$= \int_{0}^{1} \left(x^4 - x^3 \right) dx$$

$$= \left. \frac{x^5}{5} - \frac{x^4}{4} \right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

Step 2: To evaluate $\oint_C Mdx + Ndy$

$$y = x^2$$

$$dy = 2xdx$$



Along y = x^2

$$y = x, dy = dx$$
$$x = 0 \text{ to } x = 1$$
$$= \int_0^1 \left[x.x + x^2\right]dx + x^2 dx$$
$$= \int_0^1 (x^2 + x^2 + x^2)dx$$
$$= \int_0^1 3x^2 dx$$
$$= 3x\frac{3}{3}\Big]_1^0 = -1$$

∴ The required integral = $\frac{19}{20} - 1 = -\frac{1}{20}$

∴ Green's Theorem is verified

8.3. Let us sum up

So far we have seen the verification of gauss, stoke's and green's theorem. For example if we want to evaluate.

ϕ mdn+nly, we can use green's theorem depending on the problem given.

8.4. Check your progress

1. State Green's theorem
2. State Gauss theorem
3. State Stoke's theorem

8.5. Lesson End Activities

1. Verify Green's theorem in a plane with respect to $\int (x^2 - y^2)dx + 2xydy$ Where C is the boundary of the rectangle in the xoy plane bounded by the lines x = o, x = a, y = o, y = b (Ans: $2ab^2$)

2. Verify Green's Theorem for $\iint (3x^2 - 8y^2)dx + (4y - 6xy)dy$ Where C is the boundary of the region defined by the lines x = 0, y = 0, x + y = 1. (Ans: 5/3)

3. Use Green's theorem in a plane to evaluate $\int (2x-y)dx+(x-y)dy$ where C is the boundary of the circle $x^2+y^2 = a^2$ in xoy plane.

4. Use Green's theorem in a plane to evaluate $\int x^2(1+y)dx+(x^3+y^3)dy$ Where C is the square formed by x = ± 1 and y = ± 1.

5. Verify Green's theorem in a plane with respect to $\int (x^2dx - xydy)$ Where C is the boundary of the square formed by x = 0, y = 0, x = a , y = a

6. Verify Gauss divergence theorem for $\vec{f} = (x^2-yz)\vec{i}+(y^2-zx)\vec{j}+(z^2-xy)\vec{k}$ and the closed surface of the rectangular parallel piped formed by x = 0, x = 1, y = 0, y = 2, z = 0, z = 3.

7. Verify divergence theorem for $\vec{f} = 4xz\vec{i} - y\vec{j} + yz\vec{k}$ when S is a closed surface of the cube formed by x = o, x = 1, y = o, y = 1, z = 0, z = 1.

8. Use divergence theorem to evaluate $\iint_S (yz^2\vec{i}+2x^2\vec{j}+2z^2\vec{k}).\vec{ds}$ Where S is the closed surface boundary by the xoy plane and the upper half of the sphere $x^2+y^2+z^2=a^2$ above this plane

9. Use divergence theorem to evaluate $\iint_S (4x\vec{i}-2y^2\vec{j}+z^2\vec{k}).\vec{ds}$ Where S is the closed surface bounded by the cylinder $x^2+y^2=4$ and the planes z = 0 and z = 3. (April 2004)

10. Verify gauss's theorem for $\vec{f} = 2xz\vec{i}+yz\vec{j}+z^2\vec{k}$ over the upper half of the sphere $x^2+y^2+z^2=a^2$ (April 2005)

8.6. Points for discussion

1. Verify divergence theorem for $\vec{f} = (x^2-yz)\vec{i}+(y^2-zx)\vec{j}+(z^2-xy)\vec{k}$ taken over the rectangular parallellopiped $0 \le x \le a, 0 \le y \le b$ (November 2006)

2. Using gauss theorem find the value of $\int \vec{f}.\vec{n}\, ds$ where $\vec{f} = xy^2\vec{i}+yz\vec{j}+zx^2\vec{k}$ and S is the surface bounded by x = 0,x = 1, y = 0, y =2, z = 0, z = 3 (November 2005)

3. Verify gauss theorem for $\vec{f} = 4xz\vec{i} + y^2\vec{j} - yz\vec{k}$ over the cube x = 0, x = 1, y = 0, y = 1, z = 0, z = 1 (November 2000)

4. Verify Stoke's theorem for $\vec{f} = x\vec{i} + z^2\vec{j} + y^2\vec{k}$ over the surface x + y + z = 1 lying in the first octant. (November 2000)

5. Verify Stoke's theorem for $\vec{f} = (2x - y)\vec{i} - yz^2\vec{j} + y^2z\vec{k}$ Where S is the upper half of the sphere $x^2 + y^2 + z^2 = 1$ and C its boundary (April 2004)

6. Verify stoke's theorem for the function $\vec{f} = -y\vec{i} + 2yz\vec{j} + z^2\vec{k} = a^2$ and C its boundary (April 2005)

7. Verify stoke's theorem for the function $\vec{f} = y^2\vec{i} - y\vec{j} + xz\vec{k}$ and for the surface S which is upper half of the sphere $x^2 + y^2 + z^2 = a^2$ and $z \geq 0$ (November 2004 Bharathiar)

8. Verify stoke's theorem for $\vec{f} = x\vec{i} + z^2\vec{j} + y^2\vec{k}$ over the surface $x + y + z = 1$ lying in the first octant (November 2000: Bharathiyar)

9. Verify stoke's theorem for $\vec{f} = (2x - x^2)\vec{i} - (x^2 - y^2)\vec{j}$ and C is the boundary of the region enclosed by the parabolas $y^2 = x$ and $x^2 = y$ (Ans: $-\frac{3}{5}$)

10. Verify stoke's theorem to evaluate $\int \vec{f}.\vec{dr}$ where $\vec{f} = (\sin x - y)\vec{i} - \cos x\vec{j}$ and C is the boundary of the triangle. Whose vertices are (0,0), $\left(\pi/2, 0\right)$ and $\left(\pi/2, 1\right)$

Ans: $\left(\pi/4 + 2/\pi\right)$

8.7 References

1) Vector Calculus by - Namasivayam
2) Vector Calculus by - Rasinghmia aggarval
3) Vector calculus by - P. Durai Pandian
4) Vector calculus by - Chatterjee

Unit III
Lesson - 9

Fourier Series

Contents

9.0Aim and Objectives

In this lesson we are going to study about the fourier series which is a trigonometric series defined in various intervals (viz) (0,2B) (-B , B) (0,2l), (-l,l) and cosine and sine series of f(x) defined in (o,l).

9.1 Fourier Series:

A trigonometric expression of the form $f(x) = \frac{an}{2} + \sum_{n=1}^{\infty} an \cos nx + \sum_{n=1}^{\infty} bn \sin nx$ is called a Fourier expansion of f(x) where f(x) is defined in a specified internal. a_0, a_n, b_n are called Fourier Coefficients:

Model :1

Let f(x) be defined in the interval $(0,2\pi)$. Then its fourier expansion is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{--(1)}$$

Where $$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx$$

9.2. Examples

1. Determine a fourier series for f(x) = x^2 in $(0, 2\pi)$

Step 1: To find a_0

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2 dx = \frac{1}{\pi}\frac{x^3}{3}\Bigg]_0^{2\pi} = \frac{1}{\pi}.\frac{8\pi^3}{3} = \frac{8\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx dx$$

$$u = x^2 \qquad dv = \cos nx dx \qquad V_1 = -\frac{1}{n^2}\cos nx$$

$$u^1 = 2x \qquad \int dv = \int \cos nx dx \qquad V_2 = -\frac{1}{n^2}\sin nx$$

$$u^{11} = 2 \qquad V = \frac{\sin nx}{n}$$

$$\therefore \quad a_n = \frac{1}{\pi}\left[u\varphi - u^1\varphi_1 + u^{11}\varphi_2 - - - - -\right]$$

$$= \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n} - 2x\left(\frac{-1}{n^2}\cos nx\right) + 2.\left(\frac{-1}{n^2}\sin nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2}{n^2} n \cos nx - \frac{2}{n^2}\sin nx\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[\frac{2}{n^2} 2\pi \times 1\right] - [0]\right\}$$

$$= \frac{4\pi}{\pi n^2} = \frac{4}{n^2}$$

$$a_n = \frac{4}{n^2}$$

To find b_n

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nxdx$$

$$u = x^2 \qquad dv = \sin nxdx \qquad V_1 = -\frac{1}{n^2}\sin nx$$

$$u^1 = 2x \qquad \int dv = \int \sin nxdx \qquad V_2 = -\frac{1}{n^3}\sin nx$$

$$u^{11} = 2 \qquad V = -\frac{\cos nx}{n}$$

$$\therefore \quad b_n = \frac{1}{\pi}\left[uv - \int udv\right]$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{1}{n}\cos nx\right) - 2x\left(\frac{-1}{n^2}\sin nx\right) + 2.\left(\frac{1}{n^3}\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x^2\cos nx}{n} + \frac{2}{n^2}x\sin nx + \frac{2}{n^2}\cos nx\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{1}{n}(4\pi^2) + \frac{2}{n^3}\right] - \left[\frac{2}{n^3}\right]\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{4\pi^2}{n} + \frac{2}{n^3} - \frac{2}{n^3}\right\}$$

$$b_n = \frac{-4\pi}{n}$$

$\therefore$ using in (1)

$$f(x) = \frac{8\pi^2/3}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{4\pi}{n}\sin nx$$

$$= 4\pi^2/3 + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nx}{n}$$

Model 2:

Let f(x) be a function of x defined in $(-\pi, \pi)$. Then its Fourier expansion is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an\cos nx + \sum_{n=1}^{\infty} bn\sin nx$$

Where $$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nxdx$$

Example: $f(x) = x + x^2;$ $(-\pi, \pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} xdx + \int_{-\pi}^{\pi} x^2 dx\right]$$

$$= \frac{1}{\pi}\left[\left.\frac{x^2}{2}\right)_{-\pi} + \left.\frac{x^3}{3}\right)_{-\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{3}\left(\pi^3 - (-\pi^3)\right)\right]$$

$$= \frac{1}{\pi}.\frac{1}{3}.2\pi^3 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\cos nxdx$$

$u = x + x^2$ $dv = \cos nxdx$ $V_1 = -\frac{1}{n^2}\cos nx$

$u^1 = 1 + 2x$ $\int dv = \int \sin nxdx$ $V_2 = -\frac{1}{n^3}\sin nx$

$u^{11} = 2$ $V = \frac{\sin nx}{n}$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{\pi} xdx + \int_{-\pi}^{\pi} x^2 dx\right]$$

$$= \frac{1}{\pi}\left[\left.\frac{x^2}{2}\right)_{-\pi} + \left.\frac{x^3}{3}\right)_{-\pi}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{3}\left(\pi^3 - (-\pi^3)\right)\right]$$

$$= \frac{1}{\pi}.\frac{1}{3}.2\pi^3 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nxdx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\cos nxdx$$

$u = x + x^2$ $dv = \cos nxdx$ $V_1 = -\frac{1}{n^2}\cos nx$

$u^1 = 1 + 2x$ $V_2 = \frac{1}{n^3}\sin nx$

$u^{11} = 2$ $V = \frac{\sin nx}{n}$

$$\therefore \quad a_n = \frac{1}{\pi}\left[u\varphi - u^1\varphi_1 + u^{11}\varphi_2 - - - - - -\right]$$

$$= \frac{1}{\pi}\left[(x + x^2)\frac{\sin nx}{n} - (1 + 2x)\left(\frac{-1}{n^2}\cos nx\right) + 2.\left(\frac{-1}{n^3}\sin nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{(x + x^2)\frac{\sin nx}{n} + \frac{1}{n^2}(1 + 2x)\cos nx - \frac{2}{n^3}\sin nx\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[0 + \frac{1}{n^2}(1 + 2\pi)(-1)^n - 0\right] - \left[0 + \frac{1}{n^2}(1 - 2\pi)(-1)^n\right]\right.$$

$$= \frac{1}{\pi}\left\{\left[\frac{(-1)^n}{n^2}(1+2\pi -1+2\pi)\right]\right.$$

$$= \frac{(-1)^n 4\pi}{\pi n^2} = \frac{(-1)^n 4}{n^2}$$

$$\therefore a_n = \frac{(-1)^n 4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx dx$$

$u = x + x^2$ $\quad$ $dv = \sin nx dx$ $\quad$ $V_1 = -\frac{1}{n^2}\sin nx$

$u^1 = 1+2x$ $\quad$ $\int dv = \int \sin nx dx$ $\quad$ $V_2 = \frac{1}{n^3}\cos nx$

$u^{11} = 2$ $\quad$ $V = \frac{-\cos nx}{n}$

$$b_n = \frac{1}{\pi}\left[uv - \int udv\right]$$

$$= \frac{1}{\pi}\left\{(x+x^2)\left(-\frac{\cos nx}{n}\right) - (1+2x)\left(\frac{-1}{n^2}\sin nx\right) + 2.\left(\frac{1}{n^3}\cos nx\right)\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{-(x+x^2)\frac{\cos nx}{n} + \frac{1}{n^2}(1+2x)\sin nx + \frac{2}{n^3}\cos nx\right\}_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{(\pi+\pi^2)(-1)^n}{n} + 0 + \frac{2}{n^3}(-1)^n\right] - \left[-\frac{(-\pi+\pi^2)(-1)^n}{n} + 0 + \frac{2}{n^3}(-1)^n\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{(-1)^n}{n}(-\pi - \pi^2 + \pi^2 - \pi) + \frac{2}{n^3}(-1)^n - \frac{2}{n^3}(-1)^n\right\}$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n}(-2\pi)\right]$$

$$= \frac{-2\pi}{\pi n}(-1)^n = \frac{-2(-1)^n}{n}$$

$$\therefore\ f(x)=\frac{\pi^2}{3}+\sum_{n=1}^{\infty}(-1)^n\left[\frac{4}{n^2}\cos nx-\frac{2}{n}\sin nx\right]$$

Model 3: Change of Interval:

Let f(x) be a function of x defined in the interval (0, 2*l*). Then its fouriers expansion is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}an\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}bn\sin\left(\frac{n\pi x}{l}\right)$$

Where $a_0=\frac{1}{l}\int_0^{2l}f(x)dx$

$$a_n=\frac{1}{l}\int_0^{2l}f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n=\frac{1}{\pi}\int_0^{2l}f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Problem:

Determine a fourier series for

$$f(x)=\frac{x}{l};0<x<l$$

$$=\frac{2l-x}{l};\qquad l<x<2l$$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}an\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}bn\sin\left(\frac{n\pi x}{l}\right)$$

$$a_0=\frac{1}{l}\int_0^{2l}f(x)dx$$

$$=\left(\int_0^{L}f(x)dx+\int_0^{2l}f(x)dx\right)$$

$$=\frac{1}{l}\left[\left(\int_0^{L}\frac{x)}{l}dx+\int_0^{2l}\left(\frac{2l-x}{l}\right)dx\right)\right]$$

$$= \frac{1}{l^2}\left[\left(\frac{x^2}{2}\right)_0^l + \left(2lx - \frac{x^2}{2}\right)_l^{2l}\right]$$

$$= \frac{1}{l^2}\left[\frac{l^2}{2} + \left(4l^2 - \frac{4l^2}{2}\right) - (2l^2 - \frac{l^2}{2}\right]$$

$$= \frac{1}{l^2}\left[\frac{l^2}{2} + \frac{4l^2}{2} - \frac{3l^2}{2}\right]$$

$$= \frac{1}{l^2}\left[\frac{l^2}{2} + \frac{l^2}{2}\right] = \frac{1}{l^2}.\frac{2l^2}{2} = 1$$

$$a_0 = 1$$

To find a_n

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\left(\int_0^L \frac{x}{l}\cos\left(\frac{n\pi x}{l}\right)dx + \int_0^{2l}\frac{2l-x}{l}\cos\left(\frac{n\pi x}{l}\right)dx\right)\right]$$

$$= \frac{1}{l^2}\left[I_1 + I_2\right] \qquad \text{--------(1)}$$

$$I_1 = \int_0^L \frac{x}{l}\cos\frac{n\pi x}{l}dx; \qquad I_2 = \int_0^{2l}(2l-x)\cos\left(\frac{n\pi x}{l}\right)dx$$

To find I_1

Integrate by Parts

$$u = x \qquad dv = \cos\frac{n\pi x}{l}dx$$

$$u^1 = 1 \qquad \int dv = \int \cos\frac{n\pi x}{l}dx$$

$$v = \frac{1}{n\pi}\sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore I_1 = uv - \int v dx$$

$$= x.\frac{l}{n\pi}\sin\frac{n\pi x}{l}\Big]_0^l - \int_0^l \frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= -\frac{l}{n\pi}\left[\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}\right]_0^l$$

$$= -\frac{l^2}{n^2\pi^2}\left[(\cos(n\pi) - 1\right]$$

$$= -\frac{l^2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

To find I_2

$$I_2 = \int_0^{2l} (2l - x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$u = 2l - x \qquad dv = -dx$$

$$u^1 = -1 \qquad \int dv = \int \cos\frac{n\pi x}{l}\,dx$$

$$v = \frac{1}{n\pi}\sin\left(\frac{n\pi x}{l}\right)$$

$$\therefore I_1 = uv - \int v dx$$

$$= (2l - x)\frac{l}{n\pi}\sin\frac{n\pi x}{l}\Big]_l^{2l} - \int_l^{2l} \frac{l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)(-dx)$$

$$= \frac{l}{n\pi}\left[\sin\left(\frac{n\pi x}{l}\right)\right]_l^{2l}$$

$$= \frac{l^2}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{l}\right)\right]_l^{2l}$$

$$= -\frac{l^2}{n^2\pi^2}\left\{\cos 2n\pi - \cos n\pi\right\}$$

$$= -\frac{l^2}{n^2\pi^2}\left[1 - (-1)^n\right]$$

$\therefore$ using in (1)

$$a_n = \frac{1}{l^2}\left[\frac{l^2}{n^2\pi^2}((-1)^n - 1) - \frac{l^2}{n^2\pi^2}(1-(-1)^n)\right]$$

$$= \frac{1}{l^2}.\frac{l^2}{n^2\pi^2}\left[(-1)^n - 1 - 1 + (-1)^n\right]$$

$$= \frac{1}{n^2\pi^2}\left[2(-1)^n - 2\right]$$

$$a_n = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

When n is even, $a_n = 0$

When n is odd, $a_n = \frac{2}{n^2\pi^2}(-2) = \frac{-4}{n^2\pi^2}$

Step 3: To find b_n

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\left(\int_0^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right)\right]$$

$$= \frac{1}{l^2}\left[\left(\int_0^{l} x\sin\left(\frac{n\pi x}{l}\right)dx + \int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right)\right]$$

$$= \frac{1}{l^2}[I_3 + I_4] \qquad \text{--------(2)}$$

Where

$$I_3 = \int_0^{l} x\sin\left(\frac{n\pi x}{l}\right)dx; \qquad I_4 = \int_l^{2l} (2l - x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$I_3 = \int_0^{l} x\sin\left(\frac{n\pi x}{l}\right)dx$$

$$u = x \qquad dv = \sin\left(\frac{n\pi x}{l}\right)dx$$

$$u^1 = 1 \qquad \int dv = \int \sin\left(\frac{n\pi x}{l}\right)dx$$

$$du = dx \qquad v = -\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right); \quad v_1 = -\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}$$

$$I_3 = uv - u^1 v_1 + ...$$

$$= \left[x\left(-\frac{l}{n\pi}\right)\cos\left(\frac{n\pi x}{l}\right) - 1.\left(\frac{-l^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \left[-x\frac{l}{n\pi}x\cos\left(\frac{n\pi x}{l}\right) + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \left\{-\frac{l}{n^2\pi^2}\left[(l(-1)^n + 0)\right] - [0]\right\}$$

$$= \frac{-l^2}{n\pi}(-1)^n$$

$$I_4 = \int_l^{2l} (2l - x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$u = 21 - x \qquad dv = \sin\left(\frac{n\pi x}{l}\right)dx$$

$$u^1 = -1 \qquad \int dv = \int \sin\left(\frac{n\pi x}{l}\right)dx$$

$$v = -\frac{l}{n\pi}\cos\left(\frac{n\pi x}{l}\right); \quad v_1 = \frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}$$

$$I_3 = uv - u^1 v_1 + ...$$

$$= \left[-(2l - x)\frac{1}{n\pi}\cos\left(\frac{n\pi x}{l}\right) - 1.\left(\frac{-l^2}{n^2\pi^2}\right)\sin\left(\frac{n\pi x}{l}\right)\right]_l^{2l}$$

$$= \left[-\frac{l}{n\pi}(2l - x)\cos\left(\frac{n\pi x}{l}\right) - \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_l^{2l}$$

$$= \left\{ \left[-(0-0) \right] - \left[-\frac{l^2}{n^2\pi^2}\cos(n\pi) - 0 \right] \right\}$$

$$= \frac{l^2}{n\pi}\cos n\pi = \frac{l^2}{n\pi}(-1)^n$$

Using in (2)

$$b_n = \frac{1}{l^2}\left[\frac{-l^2}{n\pi}(-1)^n + \frac{l^2}{n\pi}(-1)^n \right] = 0$$

$$b_n = 0$$

$$f(x) = \frac{1}{2} - \sum_{n=1,3,5}^{\infty} \frac{4}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{1}{2} . \frac{4}{\pi^2}\sum_{1,3,5}^{\infty}\cos\left(\frac{n\pi x}{l}\right)$$

Model 4:

Let f(x) be a function of x defined the interval $(-l, l)$. Then its fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} bn\sin\left(\frac{n\pi x}{l}\right)$$

Where $$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{\pi}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Problem: Determine a fourier series for f(x) = x^2 in (-1,1)

Here l = 1

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an\cos(n\pi x) + \sum_{n=1}^{\infty} bn\sin(n\pi x)$$

$$a_0 = \int_{-1}^{1} x^2 dx = \left.\frac{x^3}{3}\right]_{1}^{-1} = \frac{1^3}{3} - \frac{(-1)^3}{3} = \frac{2}{3}$$

$$a_n = \int_{-1}^{1} x^2 \cos n\pi x dx$$

$$u = x^2 \qquad dv = \cos n\pi x dx$$

$$u^1 = 2x \qquad v = \frac{\sin n\pi x}{n\pi}$$

$$u^{11} = 2 \qquad v_1 = -\frac{1}{n^2\pi^2}\cos n\pi x; \quad v_2 = -\frac{1}{n^3\pi^3}\sin n\pi x$$

$$a_n = uv - u^1 v_1 + u^{11} v_2 ....$$

$$= \frac{x^2 \sin n\pi x}{n\pi} - 2x\left(-\frac{1}{n^2\pi^2}\cos n\pi x\right) + 2\left(-\frac{1}{n^3\pi^3}\sin n\pi x\right)\Bigg]_{-1}^{1}$$

$$= \left[\frac{x^2 \sin n\pi x}{n\pi} + \frac{2}{n^2\pi^2} x \cos n\pi x - \frac{2}{n^3\pi^3}\sin n\pi x\right]_{-1}^{1}$$

$$= \left\{\left[0 + \frac{2}{n^2\pi^2}\cos n\pi - 0\right] - \left[0 - \frac{2}{n^2\pi^2}\cos n\pi - 0\right]\right\}$$

$$= \frac{2}{n^2\pi^2}\cos n\pi + \frac{2}{n^2\pi^2}\cos n\pi$$

$$= \frac{4}{n^2\pi^2}\cos n\pi$$

$$= \frac{4}{n^2\pi^2}(-1)^n = \frac{(-1)^n 4}{n^2\pi^2}$$

To find b_n:

$$b_n = \int_{-1}^{1} f(x) \sin n\pi x dx$$

$$= \int_{-1}^{1} x^2 \sin n\pi x dx$$

$$u = x^2 \qquad dv = \sin n\pi x dx$$

$$u^1 = 2x \qquad v = -\frac{1}{n\pi}\cos n\pi x$$

$$u^{11} = 2 \qquad v_1 = -\frac{1}{n^2\pi^2}\sin n\pi x; \quad v_2 = \frac{1}{n^3\pi^3}\cos n\pi x$$

$$\therefore b_n = u\varphi - u^1\varphi_1 + u^{11}\varphi_2 + u^{111}\varphi_3 + \ldots\ldots$$

$$= x^2\left(-\frac{1}{n\pi}\cos n\pi x\right) - 2x\left(-\frac{1}{n^2\pi^2}\sin n\pi x\right) + 2\left(\frac{1}{n^3\pi^3}\cos n\pi x\right)\Bigg]_{-1}^{1}$$

$$= -\frac{1}{n\pi}x^2\cos n\pi x + \frac{2}{n^2\pi^2}x\sin n\pi x + \frac{2}{n^3\pi^3}\cos n\pi x\Bigg]_{-1}^{1}$$

$$= \left\{\left[-\frac{1}{n\pi}(-1)^n + 0 + \frac{2}{n^3\pi^3}(-1)^n\right] - \left[-\frac{1}{n\pi}\cos n\pi + 0 + \frac{2(-1)^n}{n^3\pi^3}\right]\right\}$$

$$= -\frac{1}{n\pi}(-1) + (-1)^n\frac{2}{n^3\pi^3} + \frac{1}{n\pi}\cos n\pi - \frac{2}{n^3\pi^3}(-1)^n$$

$$b_n = 0$$

$$\therefore \qquad f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2\pi^2}\cos(n\pi x)$$

Model 5: Cosine series and sine series

a). Let f(x) be a function defined in the interval (0,*l*) then its cosine series is defined

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} an\cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

b. Its sine series is $f(x) = \sum_{n=1}^{\infty} bn\sin\left(\frac{n\pi x}{l}\right)$

Where $a_0 = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Examples

1. Determine a cosine series for $f(x) = (x-1)^2; \quad in \quad 0 < x < 1.$

Here $l = 1$

$$f(x) = \frac{a_0}{2} + \sum_{1}^{\infty} an \cos n\pi x$$

$$a_0 = 2\int_0^1 f(x)dx = 2\int_0^1 (x-1)^2\, dx$$

$$= 2\left.\frac{(x-1)^3}{3}\right]_0^1$$

$$= \frac{2}{3}(0+1) = \frac{2}{3}$$

$$a_n = 2\int_0^l f(x)\cos n\pi x dx$$

$$= 2\int_0^l (x-1)^2 \cos n\pi x dx$$

$$u = (x-1)^2 \qquad dv = \cos n\pi x dx$$

$$u^1 = 2(x-1) \qquad \int dv = \int \cos n\pi x dx$$

$$u^{11} = 2 \qquad v = \frac{\sin n\pi x}{n\pi}$$

$$v_1 = -\frac{1}{n^2\pi^2}\cos n\pi x$$

$$v_2 = -\frac{1}{n^3\pi 3}\sin n\pi x$$

$$a_n = 2\left[u\varphi - u^1\varphi_1 + u^{11}\varphi_2 \ldots\ldots\ldots\ldots\right]$$

$$= 2\left[(x-1)^2\frac{\sin n\pi x}{n\pi} - (2)(n-1)\left(-\frac{1}{n^2\pi^2}\cos n\pi x\right) + 2\left(-\frac{1}{n^3\pi^3}\sin n\pi x\right)\right]_0^1$$

$$= 2\left[(x-1)^2\frac{\sin n\pi x}{n\pi} + \frac{2}{n^2\pi^2}(n-1)\cos n\pi x - \frac{2}{n^3\pi^3}\sin n\pi x\right]_0^1$$

$$= 2\left\{[0+0+-0] - \left[-\frac{2}{n^2\pi^2}\times 1 - 0\right]\right\}$$

$$a_n = \frac{4}{n^2\pi^2}$$

$$\therefore\ f(x) = \frac{1}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\cos n\pi x$$

b) Sine series

$$f(x) = (x-1)^2; \quad (0,1).$$

$$f(x) = \sum_{1}^{\infty} bn \sin\left(\frac{n\pi x}{l}\right);$$

Here $l = 1$

$$\therefore \quad b_n = 2\int_0^1 (x-1)^2 \sin(n\pi x) dx$$

$$u = (x-1)^2 \qquad dv = \sin n\pi x dx$$

$$u^1 = 2(x-1) \qquad \int dv = \int \sin n\pi x dx$$

$$u^{11} = 2 \qquad v = \frac{-\cos n\pi x}{n\pi}$$

$$v_1 = -\frac{1}{n^2\pi^2} \sin n\pi x$$

$$v_2 = \frac{1}{n^3\pi 3} \cos n\pi x$$

$$b_n = 2[u\varphi - u^1\varphi_1 + u^{11}\varphi_2 ....]$$

$$= 2\left[(x-1)^2\left(-\frac{1}{n\pi}\cos n\pi x\right) - 2(n-1)\left(-\frac{1}{n^2\pi^2}\sin n\pi x\right) + 2\left(\frac{1}{n^3\pi^3}\cos n\pi x\right)\right]$$

$$= 2\left\{-\frac{1}{n\pi}(x-1)^2 \cos n\pi x + \frac{2(n-1)}{n^2\pi^2}\sin n\pi x + \frac{2}{n^3\pi^3}\cos n\pi x\right\}_0^1$$

$$= 2\left\{\left[0 + 0 + \frac{2}{n^3\pi^3}\cos n\pi\right] - \left[\frac{1}{n\pi} + 0 + \frac{2}{n^3\pi^3}\right]\right\}$$

$$= 2\left[\frac{2}{n^3\pi^3}(-1)^n - \frac{2}{n^3\pi^3}\right]$$

$$b_n = \frac{4}{n^3\pi^3}\left[(-1)^n - 1\right]$$

When n is even, $b_n = 0$

When n is odd, $b_n = \frac{-8}{n^3\pi^3}$

$$\therefore f(x) = -\frac{8}{\pi^3} + \sum_{n=1,3,5,....}^{\infty} \frac{1}{n^3} \sin n\pi x$$

9.3. Let us sum up

We studied so far in finding fourier series in the intervals (0,2B),(-B,B), (0,2l), (-l,l) etc.,

Due care should be taken in finding a_n and b_n while for use the formula "Bernonllis formula on Integration by parts."

9.4. Check your progress

(1) Find a_0 if f(x) =x; (0,2B)

(2) Find b_n if f(x) = x2 (-B,B)

(3) Find an if f(x) = a, (0,2l) where a is a constant

9.5. Lesson End Activities

1. If $f(x) = 1, \; 0 < x < \pi,$
   $= -1, \pi < x < 2\pi,$ prove that
   $$f(x) = \frac{4}{\pi^3} \sum_{n=1,3,5,....}^{\infty} \frac{\sin nx}{n}$$
2. If $f(x) = x(2\pi - n),$ show that
   $$f(x) = \frac{2\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx; \quad (0,2\pi)$$
3. If $f(x) = R, \; 0 < x < \pi,$
   $= -R, \pi < x < 2\pi,$

   Prove that
   $$f(x) = \frac{4n}{\pi} \sum_{n=1,3,5,....}^{\infty} \frac{1}{n} \sin nx$$
4. If $f(x) = -1 + x, \; -\pi < x < 0$
   $= 1 + x, \; 0 < x < \pi$

   Prove that
   $$f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[1 - (-1)^n (1+\pi)\right] \sin nx$$

9.6. Points for discussion

1. $f(x) = -\pi, \; -\pi < x < 0$
   $= \pi, \; 0 < x < \pi$

   Show that
   $$f(x) = \frac{-\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right)$$

2. $f(x) = 2, \quad -\pi < x < 0$
   $= 4, \; 0 < x < \pi$

   Prove that

$$f(x) = 3 + \frac{4}{\pi} \sum_{n=1,3,5....}^{\infty} \frac{\sin nx}{n}$$

9.7 References

1. Fourier series by S. Narayanan T.K.M. Pillai

Lesson 10

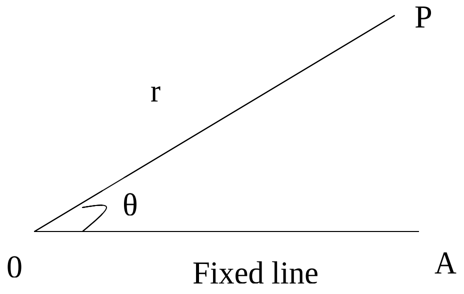
Contents

In this lesson we are going to learn about the concept of the polar coordinates, converting the castesian system to polar system, polar of a straight line, circle, a concern.

10.1 Analytical Geometry

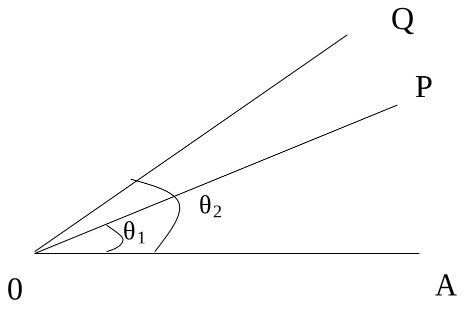
10.1.1. Polar coordinates:

We shall study another type of coordinates called polar coordinates to represent plane curves. By this system of coordinates, any point P is specified by its distance from a fixed point O and the angle θ made by the join of P and O with a fixed line out.



The point P is represented by the polar coordinate P(r, θ); OP = r is called the radius vector and θ is called the vectorial angle. The radius vector is considered positive if measured from O along the line bounding the vectorial angle and negative in the opposite direction.

10.1.2. Distance between the points (r_1, θ_1) and (r_2, θ_2).



Let the points be P (r_1, θ_1) and Q (r_2, θ_2).

OP = r_1; OQ=r_2; $AOP = \theta_1$; $AOQ = \theta_2$

$POQ = \vartheta_2 - \vartheta_1$

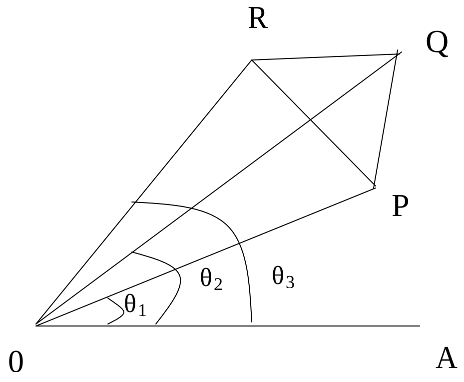
In Δ OPQ, we have

$$OP^2 + 0Q^2 - 2OP.OQ\cos POQ = PQ^2 \text{ (Cosine formula)}$$

$$r_1{}^2 + r_2{}^2 - 2r_1r_2\cos(\vartheta_2 - \vartheta_1) = P\vartheta^2$$

$$\therefore PQ = \sqrt{r_1{}^2 + r_2{}^2 - 2r_1r_2\cos(\vartheta_2 - \vartheta_1)}$$

10.1.3. Area of triangle in polar coordinates



Let O be the pole:Let OA be the initial lline

Let PQR be the triangle whose vertices are P (r_1, θ_1) , Q (r_2, θ_2) and R (r_3, θ_3)

Let OP = r_1; OQ=r_2; QR=r_3

$A\hat{O}P = \theta_1$; $A\hat{O}Q = \theta_2$ $A\hat{O}Q = \theta_3$

In Δ OPQ, $POQ = \vartheta_2 - \vartheta_1$

$$\therefore \quad \text{Area of } \Delta OPQ = \frac{1}{2}.OP.OQ.\sin(\theta_2 - \theta_1)$$
$$= \frac{1}{2}.r_1.r_2.\sin(\theta_2 - \theta_1)$$

In Δ QOR, $QOR = \vartheta_3 - \vartheta_2$

$$\therefore \quad \text{Area of } \Delta QOR = \frac{1}{2}.OQ.OR.\sin QOR$$
$$= \frac{1}{2}.r_2.r_3.\sin(\theta_3 - \theta_2)$$

In Δ OPR, $POR = \vartheta_3 - \vartheta_1$

$$\therefore \quad \text{Area of } \Delta OPR = \frac{1}{2}.OP.OR.\sin POR$$
$$= \frac{1}{2}.r_1.r_3.\sin(\theta_3 - \theta_2)$$

$$\therefore \quad \text{Area of } \Delta PQR =$$

= Area of Δ OPQ + Area of Δ QOP – Area of Δ OPR

$$= \frac{1}{2}.r_1.r_2.\sin(\theta_2 - \theta_1) + \frac{1}{2}.r_2.r_3.\sin(\theta_3 - \theta_2) - \frac{1}{2}.r_1.r_3.\sin(\theta_3 - \theta_1)$$

$$= \frac{1}{2}.\left[r_1.r_2.\sin(\theta_2 - \theta_1) + .r_2.r_3.\sin(\theta_3 - \theta_2) - .r_1.r_3.\sin(\theta_3 - \theta_1)\right]$$

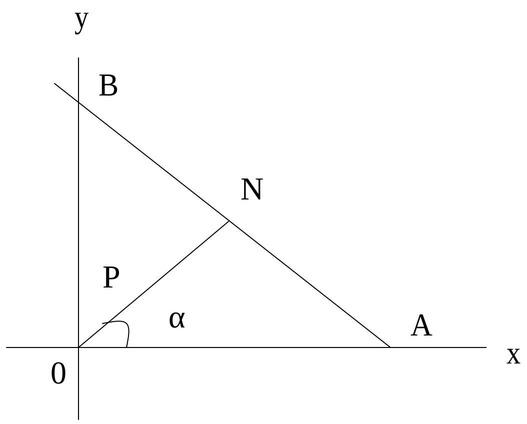
10.1.4. Conversion of polar coordinates into Cartesian coordinates

Rule: To convert the Cartesian coordinate (x,y) into polar coordinate.

Put x = r cos θ; y = r sin θ

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Equation of a straight in polar coordinates



Continue 110 pages:

Consider a straight line AB. Draw a perpendicular from the origin to the line B.

Let this perpendicular make an angle α with the x axis.

Then the equation of the straight line AB is

$x \cos \alpha + y \sin \alpha = \rho$ ----(1)

Taking the origin as the pole and the x axis as the initial line

Put $x = r \cos \theta$, $y = r \sin \theta$ in (1)

$$r \cos\theta \cos\alpha + r \sin\theta \sin\alpha = p$$

$$r[\cos\theta \cos\alpha + \sin\theta \sin\alpha] = p$$

$$r \cos(\theta - \alpha) = p$$

This is the polar equation of a straight line.

Equation of a straight line not passing through the pole.

Consider the equation of the straight line as ax+by+c = 0 ----(1)

Transforming into polar coordinates

Put $x = r \cos \theta$, $y = r \sin \theta$ in (1)

$$ar \cos\theta + br \sin\theta + c = 0$$

$$a \cos\theta + b \sin\theta + \frac{c}{r} = 0$$

$a\cos\theta + b\sin\theta = -\frac{c}{r} = K$ where $K = -\frac{c}{r}$

Which is the equation of the line not passing through the pole.

The equation of the line passing through the pole is $\theta = a$ constant.

Parallel lines:

If the equation of the line is

$a\cos\theta + b\sin\theta = K$, then the equation of the line parallel to this is

$$a\cos\theta + b\sin\theta = K'$$

Perpendicular lines:

Consider the perpendicular lines

Ax+By+C = 0 ----(1) and Bx-Ay-C' = 0 ---(2)

Put x = r cosθ, y = r sin θ in (1) & (2)

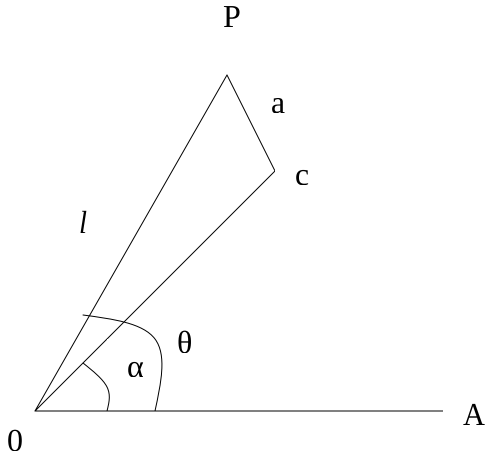
We get

A cosθ + B sinθ = K ----(1)

B cosθ – A sinθ = K' ----(2)

We observe that the line perpendicular to (1) is obtained by replacing θ by $\frac{\pi}{2} + \theta$ and K by K'.

Equation of a circle in polar coordinates



Let 0 be the pole and OA be the initial line. Let C be the centre of the circle of radius a units.

Let P be any point on the circle and OP = r; $A\hat{O}P = \vartheta$
Let $A\hat{O}P = \alpha$; $let \quad OC = c$
$\therefore COP = \theta - \alpha$

In ΔOCP

$$CP^2 = OC^2 + OP^2 - 2OC.OC.\cos(COP)$$
$$a^2 = c^2 + r^2 - 2cr\cos(\theta - \alpha) \qquad \text{-----(1)}$$

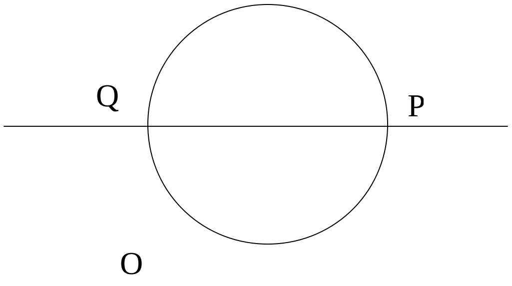
Cor1: If the pole lies on the circumference on the circle, then c = a

$\therefore$ using in (1)

$$a^2 = a^2 + r^2 - 2ar\cos(\theta - \alpha)$$
$$\therefore \quad r^2 = 2ar\cos(\theta - \alpha)$$
$$\therefore \quad r = 2a\cos(\theta - \alpha) \qquad \text{----(2)}$$

10.1.5. Equation of chord joining the points $P(\theta_1) and P(\theta_2)$ of the circle r = 2acosθ

Proof:



r=2acosθ

Consider the equation of the circle where the pole lies on it and the initial line passes through the centre of the circle.

The equation of the circle is r = 2acosθ --(1)

The points $P(\theta_1) and P(\theta_2)$ lie in the circle (1)

∴ the radii vectors of P and Q are $2a\cos\theta_1$, and $2a\cos\theta_2$ respectively

∴ $P(2a\cos\theta_1,\theta_1)$ *and* $Q(2a\cos\theta_2,\theta_2)$

Let the equation of the straight line PQ be

$$p = r\cos(\theta-\alpha) \qquad \text{----(2)}$$

∴ The points P,Q lie on (2)

∴ $$p = 2a\cos\theta_1\cos(\theta_1-\alpha) \qquad \text{----(3)}$$

$$p = 2a\cos\theta_2\cos(\theta_2-\alpha)$$

∴ $$2a\cos\theta_1\cos(\theta_1-\alpha) = 2a\cos\theta_2\cos(\theta_2-\alpha)$$

$$2\cos\theta_1\cos(\theta_1-\alpha) = 2\cos\theta_2\cos(\theta_2-\alpha)$$

But $2\cos A\cos B = \cos(A+B)+\cos(A-B)$

∴ $$\cos(\theta_1+\theta_1-\alpha)+\cos\alpha = \cos(\theta_2+\theta_2-\alpha)+\cos\alpha$$

$$\cos(2\theta_1-\alpha) = \cos(2\theta_2-\alpha)$$

∴ $$2\theta_1-\alpha = \pm 2\theta_2-\alpha$$

∴ $\theta_1 \neq \theta_2$ Taking negative sign $2\theta_1-\alpha = -(2\theta_2-\alpha)$

$$2\theta_1-\alpha = -2\theta_2+\alpha$$

$$2\theta_1+2\theta_2 = 2\alpha$$

$$\theta_1+\theta_2 = \alpha$$

Sub in (3)

$$p = 2a\cos\theta_1\cos(\theta_1-\theta_1-\theta_2)$$

$$= 2a\cos\theta_1\cos(-\theta_2)$$

$$p = 2a\cos\theta_1\cos\theta_2$$

Substitute the value of ** and α in (2) the equation of PQ is

$$2a\cos\theta_1\cos\theta_2 = r\cos(\theta-\theta_1-\theta_2) \qquad \text{---------(A)}$$

Cor 1: To find the equation of the tangent at 'α'

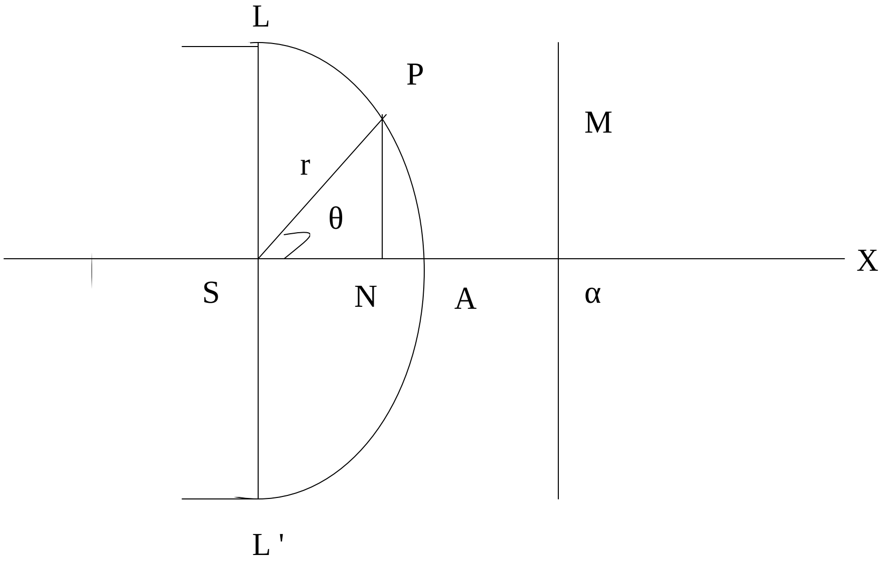
Put $\theta_1=\theta_2=\alpha$ in (A)

$$2a\cos\alpha.\cos\alpha = r\cos(\theta-2\alpha)$$

$$2a\cos^2\alpha = r\cos(\theta-2\alpha)$$

$$\therefore \qquad r\cos(\theta-2\alpha) = 2a\cos^2\alpha$$

10.1.6.Polar equation of conic



Let S be the focus and SX be the initial line

Let XM be the directrix. Let e be the eccentricity of the conic

Draw SX perpendicular to the directrix..

Let P be any point on the conic

Let P be (r,θ)

Let SP = r; XSP = θ

Draw PM perpendicular the directrix.

PN perpendicular the initial line.

Let LSL' be the latus rectum of the conic.

L is a point on the conic.

$\therefore$ By the definition of the conic,

$$\frac{SL}{SX} = e$$

$$SL = e.SX$$

But $SL = l$

$$\therefore \quad l = e.SX$$

$$SX = \frac{l}{e}$$

Also P is a point on the conic

$$\therefore \frac{SP}{PM} = e$$

$$SP = e.PM$$

$$= e.NX$$

$$= e(SX - SN) \qquad \text{---(1)}$$

But $\quad SX = \frac{l}{e}$

In $\quad \Delta SPN, \quad \cos\theta = \frac{SN}{SP}$

$$\cos\theta = \frac{SN}{r}$$

$$\therefore SN = r\cos\theta$$

$\therefore$ using in (1)

$$r = e\left(\frac{l}{e} - r\cos\theta\right)$$

$$r = l - er\cos\theta$$

$$r + er\cos\theta = l$$

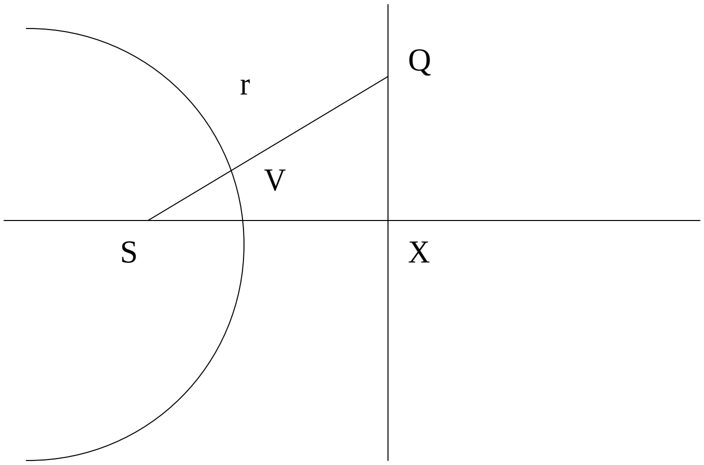
$$r(1 + e\cos\theta) = l$$

$$\frac{l}{r} = 1 + e\cos\theta$$

Note: Let SX makes an angle α with the initial line SA, then the radius vector SP makes an angle $\theta - \alpha$ with the initial line

$\therefore$ The equation of the conic in this case is $\frac{l}{r} = 1 + e\cos(\theta - \alpha)$

Note 2: Directrix corresponding to the poles.



Let Q (r, θ) be any point on the directrix.

$\therefore$ In $\Delta SQX;$ $\cos\theta = \frac{SX}{SQ}$

$$\therefore SX = SQ\cos\theta$$

$$= r\cos\theta$$

But $SX = \frac{l}{e}$

$$\therefore \frac{l}{e} = r\cos\theta$$

$$\therefore \frac{l}{r} = e\cos\theta$$

Note 3: Consider the equation of the conic $\frac{l}{r} = e\cos\theta + e\cos(\theta - \alpha)$

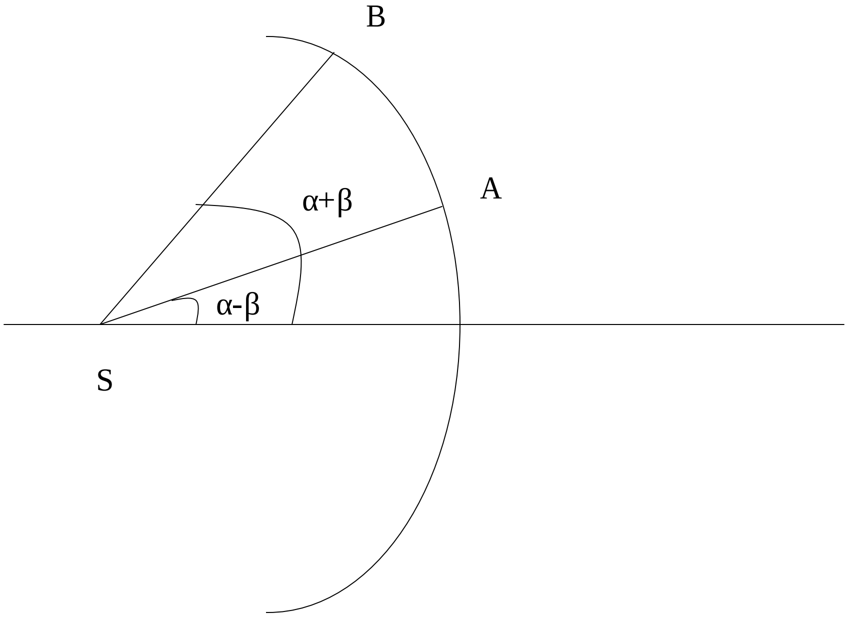
$\therefore$ Equating of the corresponding directrix is $\frac{l}{r} = e\cos(\theta - \alpha)$

10.1.7.Equation of the chord joining the points $A(\alpha - \beta)$ and $B(\alpha + \beta)$ of the conic

$$\frac{l}{r} = 1 + e\cos\theta$$

Solution: The equation of the conic is $\frac{l}{r} = 1 + e\cos\theta$ (1)

Let A and B be two points on the conic (1) whose vectorial angles are 'α-β' and 'α+β' respectively.



The equation of any chord not passing through the pole is

$$\frac{l}{r} = A'\cos\theta + B'\cos\theta \qquad \text{-----(2)}$$

The points A(SA, α-β) and B (SB, α+β) lie on (1) and (2)

$$\therefore \quad \frac{l}{SA} = 1 + e\cos(\alpha - \beta) \qquad \text{----(2)}$$

$$\frac{l}{SB} = 1 + e\cos(\alpha + \beta) \qquad \text{----(3)}$$

Also

$$\frac{l}{SA} = A'\cos(\alpha - \beta) + B'\sin(\alpha - \beta) \qquad \text{----(4)}$$

$$\frac{l}{SB} = A'\cos(\alpha + \beta) + B'\sin(\alpha + \beta) \qquad \text{----(5)}$$

From (2) and (3)

$$1 + e\cos(\alpha - \beta) = A'\cos(\alpha - \beta) + B'\sin(\alpha - \beta)$$

$$\therefore (A' - e)\cos(\alpha - \beta) + B'\sin(\alpha - \beta) = 1 \qquad \text{----(6)}$$

From (2) and (5)

$$1 + e\cos(\alpha + \beta) = A'\cos(\alpha + \beta) + B'\sin(\alpha + \beta)$$

$$\therefore (A' - e)\cos(\alpha + \beta) + B'\sin(\alpha + \beta) = 1 \qquad \text{----(7)}$$

Solve (6) and (7)

(6) X sin (α+β) given

$$(A' - e)\cos(\alpha - \beta)\sin(\alpha + \beta) + B'\sin(\alpha - \beta)\sin(\alpha + \beta) = \sin(\alpha + \beta)$$

(7) X sin (α-β) given

$$(A' - e)\cos(\alpha + \beta)\sin(\alpha - \beta) + B'\sin(\alpha + \beta)\sin(\alpha - \beta) = \sin(\alpha - \beta)$$

Subtracting

$$(A' - e)\left[\sin(\alpha + \beta)\cos(\alpha - \beta) - \cos(\alpha + \beta)\sin(\alpha - \beta)\right] = \sin(\alpha + \beta) - \sin(\alpha - \beta)$$

$$(A' - e)\left[\sin(\alpha + \beta - \alpha + \beta)\right] = 2\cos\alpha\sin\beta$$

$$\therefore A' - e = \frac{2\cos\alpha\sin\beta}{\sin 2\beta}$$

$$= \frac{2\cos\alpha\,\sin\beta}{2\sin\beta\,\cos\beta}$$

$$A' - e = \cos\alpha\,\sec\beta$$

To find B'

(6) X cos (α+β) given

$$(A' - e)\cos(\alpha - \beta)\cos(\alpha + \beta) + B'\sin(\alpha - \beta)\cos(\alpha + \beta) = \cos(\alpha + \beta)$$

(7) X cos (α-β) given

$$(A' - e)\cos(\alpha + \beta)\cos(\alpha - \beta) + B'\sin(\alpha + \beta)\cos(\alpha - \beta) = \cos(\alpha - \beta)$$

Subtracting

$$B'\left[\sin(\alpha - \beta)\cos(\alpha + \beta) - \sin(\alpha + \beta)\cos(\alpha - \beta)\right] = \cos(\alpha + \beta) - \cos(\alpha - \beta)$$

$$B'\left[\sin\left(\alpha - \beta - \alpha - \beta\right)\right] = -2\sin\alpha\,\sin\beta$$

$$B'\sin(-2\beta) = -2\sin\alpha\,\sin\beta$$

$$-B'\sin 2\beta = -2\sin\alpha\,\sin\beta$$

$$B' = \frac{2\sin\alpha\,\sin\beta}{2\sin\beta\,\cos\beta}$$

$$B' = \sin\alpha\,\sec\beta$$

$$\therefore \qquad A' = e + \cos\alpha\,\sec\beta$$

Substitute A', B' in (2)

Equation of chord AB is

$$\frac{l}{r} = (e + \cos\alpha\,\sec\beta)\cos\theta + \sin\alpha\,\sec\beta\,\sin\theta$$

$$= e\cos\theta + \cos\alpha\,\sec\beta\,\cos\theta + \sin\alpha\,\sec\beta\,\sin\theta$$

$$= e\cos\theta + \sec\beta\left[\cos\theta\cos\alpha + \sin\theta\sin\alpha\right]$$

$$\frac{l}{r} = e\cos\theta + \sec\beta\cos(\theta - \alpha) \qquad \text{-----(A)}$$

Cor 1:

When $\beta = 0$, the points A and B coincide. Chord AB becomes the tangent at 'α'

$\therefore$ Put $\beta = 0$ in (A)

Equation of the tangent at 'α' is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha)$$

BW Equation of the normal at 'α' on the conic $\frac{l}{r} = 1 + e\cos\theta$

Let the equation of the conic be

$$\frac{l}{r} = 1 + e\cos\theta \qquad \text{---(1)}$$

Equation of the tangent at 'α' is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \qquad \text{---(2)}$$

$\therefore$ The equation of the line perpendicular to the line (2) is

$$\frac{R}{r} = e\cos\left(\frac{\pi}{2} + \theta\right) + \cos\left(\frac{\pi}{2} + \theta - \alpha\right)$$

$$\frac{R}{r} = -e\sin\theta - e\sin(\theta - \alpha) \qquad \text{---(3)}$$

If this is the normal at P(α), then P (SP, α) lies on (3)

$$\frac{R}{SP} = -e\sin\alpha \qquad \text{---(4)}$$

P (SP, α) lies on the conic.

$$\therefore \qquad \frac{l}{SP} = 1 + e\cos\alpha$$

$$SP = \frac{l}{1 + e\cos\alpha}$$

Using in (4)

$$R = \frac{-le\sin\alpha}{1 + e\cos\alpha}$$

Using in (3)

Equation of the normal at 'α'.

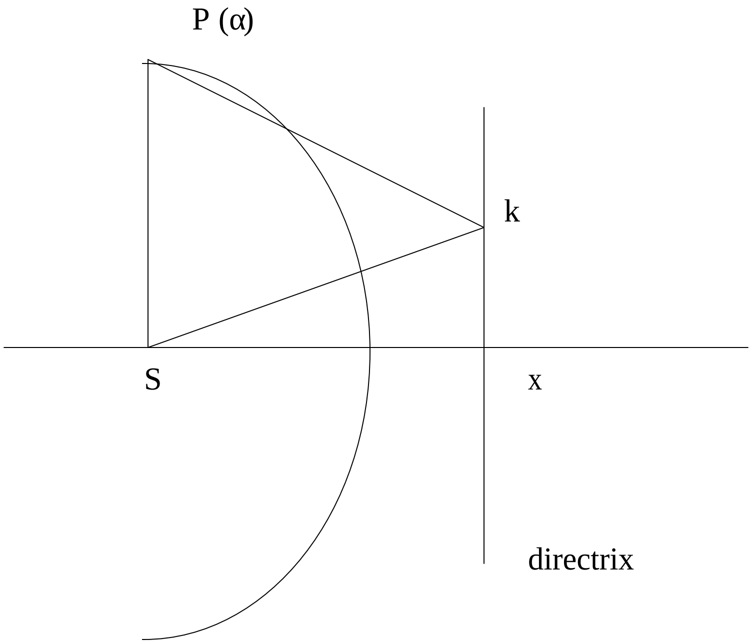
$$\frac{-le\sin\alpha}{1 + e\cos\alpha} . \frac{1}{r} = -e\sin\theta - e\sin(\theta - \alpha)$$

$$\therefore \frac{-le\sin\alpha}{r(1 + e\cos\alpha)} = e\sin\theta + e\sin(\theta - \alpha)$$

10.1.8. Properties

1. If the tangent at P to a Conic meets the directrix at K, prove that $K\hat{S}P = 90^0$

Solution :



Let the equation of the conic be

$$\frac{l}{r} = 1 + e\cos\theta \qquad \text{---(1)}$$

Let P be any point on it whose vectorial angle is α

$\therefore$ The equation of the tangent at P(α) is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \qquad \text{---(2)}$$

The equation of the directrix is

$$\frac{l}{r} = e\cos\theta \qquad \text{---(3)}$$

Solve (2) and (3)

$$e\cos\theta = e\cos\theta + \cos(\theta - \alpha)$$

$$\therefore \quad \cos(\theta - \alpha) = 0$$

$$\theta - \alpha = \pm\frac{\pi}{2}$$

$$\therefore \theta = \alpha \pm \frac{\pi}{2}$$

$$\therefore KSP = ZSP = \quad ZSK$$

$$= \alpha - \left(\alpha \pm \frac{\pi}{2}\right)$$

$$K\hat{S}P = \pm\frac{\pi}{2}$$

2. Prove that the tangents at the extremities of any focal chord of a conic $\frac{l}{r} = 1 + e\cos\theta$ intersect on the corresponding directrix.

Proof: Equation of the conic is $\frac{l}{r} = 1 + e\cos\theta$ ---(1)

Let PQ be a focal chord of (1)

Let P be (SP, α). Then Q is (SQ, $\pi + \alpha$)

$\therefore$ The tangents at P,Q is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \qquad \text{---(2)}$$

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha - \pi) \qquad \text{---(3)}$$

Let (r_1, θ_1) be the point of intersection of the tangents at P,Q

$$\therefore \quad \frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha)$$

and $$\frac{l}{r_1} = e\cos\theta_1 + \cos(\theta_1 - \alpha - \pi)$$

$$= e\cos\theta_1 + \cos(\pi + \alpha - \theta_1)$$

$$= e\cos\theta_1 - \cos(\theta_1 - \alpha)$$

$$\therefore \quad Add. \frac{2l}{r_1} = 2e\cos\theta_1$$

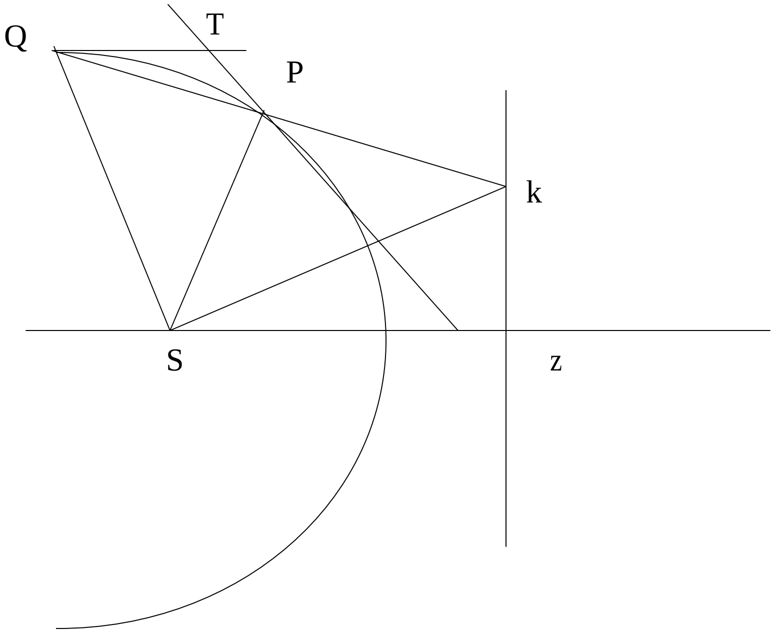
$$\therefore \quad \frac{l}{r_1} = 2e\cos\theta_1,$$

$\therefore$ The locus (r_1, θ_1) is $\frac{l}{r} = e\cos\theta$ which is the directrix

3. If the tangents at P and Q on the conic $\frac{l}{r} = 1 + e\cos\theta$ meet at T, then prove the following

a. ST bisects $P\hat{S}Q$
b. If PQ intersects the directrix at K, than $T\hat{S}K = 90^0$
c. ST^2 = SP.SQ of the conic in a parabola

Solution :



Let the equation of the conic be $\frac{l}{r} = 1 + e\cos\theta$ ----(1)

Let P,Q be the points with vectorial angles α, β respectively. Then equations of the tangent at P(α) and Q(β) are

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \qquad \text{---(1)}$$

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \beta) \qquad \text{---(2)}$$

Solve (1) and (2)

$e\cos\theta + \cos(\theta - \alpha) = e\cos\theta + \cos(\theta - \beta)$

$\therefore \quad \cos(\theta - \alpha) = \cos(\theta - \beta)$

$(\theta - \alpha) = \pm(\theta - \beta)$

Taking –ve, sign, $\theta - \alpha = -\theta + \beta$

$2\theta = \alpha + \beta$

$$\theta = \frac{\alpha + \beta}{2}$$

$$\therefore \qquad ZST = \frac{\alpha + \beta}{2}$$

Proof of (a): $ZSP = \alpha$; $\quad ZSQ = \beta$, $\quad ZST = \frac{\alpha + \beta}{2}$

$$\therefore \qquad P\hat{S}T = Z\hat{S}T - Z\hat{S}P = \frac{\alpha + \beta}{2} - \alpha$$

$$= \frac{\beta - \alpha}{2}$$

$$T\hat{S}Q = Z\hat{S}Q - Z\hat{S}T$$

$$= \beta - \left(\frac{\alpha + \beta}{2}\right)$$

$$= \frac{2\beta - \alpha - \beta}{2} = \frac{\beta - \alpha}{2}$$

$$\therefore \qquad P\hat{S}T = T\hat{S}Q$$

Is ST bisects $P\hat{S}Q$

b) The equation of PQ is

$$\frac{l}{r} = e\cos\theta + \sec\left(\frac{\alpha - \beta}{2}\right)\cos\left(\theta - \overline{\frac{\alpha + \beta}{2}}\right) \qquad \text{---(1)}$$

Equation of the directrix is $\frac{l}{r} = e\cos\theta$ ---(2)

Let the chord PQ intersect the directrix at K.

$\therefore$ solve (1) and (2)

$$e\cos\theta + \sec\left(\frac{\alpha - \beta}{2}\right)\cos\left(\theta - \overline{\frac{\alpha + \beta}{2}}\right) = e\cos\theta$$

$$\sec\left(\frac{\alpha - \beta}{2}\right)\cos\left(\theta - \overline{\frac{\alpha + \beta}{2}}\right) = 0$$

$$\therefore \qquad \cos\left(\theta - \frac{\overline{\alpha+\beta}}{2}\right) = 0$$

$$\therefore \qquad \theta - \frac{\overline{\alpha+\beta}}{2} = \pm\frac{\pi}{2}$$

$$\therefore \qquad \theta = \frac{\alpha+\beta}{2} \pm \frac{\pi}{2}$$

$$LSK \quad = \frac{\alpha+\beta}{2} \pm \frac{\pi}{2}$$

$$K\hat{S}T \quad = Z\hat{S}T - Z\hat{S}K$$

$$= \frac{\alpha+\beta}{2} - \left(\frac{\alpha+\beta}{2} \pm \frac{\pi}{2}\right) = \pm\frac{\pi}{2}$$

c) $\therefore$ The conic is a parabola, e = 1

$\therefore$ Equation of the tangent at ‘α’ is

$$\frac{l}{r} = e\cos\theta + \cos(\theta - \alpha) \qquad\qquad \text{---(1)}$$

T is $\left(ST, \frac{\alpha+\beta}{2}\right)$

The point T lies on (1)

$$\therefore \frac{l}{ST} = \cos\left(\frac{\alpha+\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2} - \alpha\right)$$

$$= \cos\left(\frac{\alpha+\beta}{2}\right) + \cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\frac{l}{ST} = 2\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)$$

$$\therefore ST = \frac{l}{2\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\beta}{2}\right)}$$

But P, Q lis on $\frac{l}{r} = 1 + \cos\theta$

$$\therefore \frac{l}{SP} = 1+\cos\alpha; \qquad \frac{1}{SQ} = 1+\cos\beta$$

$$\therefore SP = \frac{l}{1+\cos\alpha}; \qquad SQ = \frac{l}{1+\cos\beta}$$

$$SP.SQ = \frac{l^2}{(1+\cos\alpha)(1+\cos\beta)}$$

$$= \frac{l^2}{2\cos^2\frac{\alpha}{2}\,2\cos^2\frac{\beta}{2}}$$

$$= \frac{l^2}{4\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}}$$

$$= \left[\frac{l^2}{2\cos\frac{\alpha}{2}\cos\frac{\beta}{2}}\right]^2$$

$$= ST^2$$
$$\therefore ST^2 = SP.SQ$$

10.2. Examples

10.3. Let us sum up

We have studied so far hints to find the equation straight lines, circle, conic in polar coordinates.

10.4. Check your progress

(1) Find the equating the differ box to the conic l/r=1+e cosθ

(2) Find the eccentricity of the cosθ conic

l/r= 2+4 cosv

10.5. Lesson End Activities

1) If PSQ is a focal chord of the conic l/r=1+e cosθ, with the focus at

Prove that $\frac{1}{s}+\frac{1}{p}\frac{1}{s}=\frac{2}{q}$

2) If a chord of the conic l/r=1+ecos θ subtends a right angle at the focus

Prove that $\left(\frac{1}{s}-\frac{1}{p}\right)^2+\left(\frac{1}{s}-\frac{1}{q}\right)^2=\frac{e^2}{e^2}$

10.6. Points for discussion

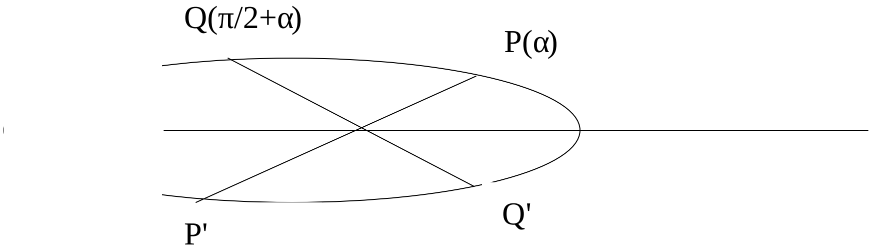
(1) If the normal at L, one of the extremities of the latus sectum of the conic l/r=1+e cosθ mees the curve again in, Show that

$$SQ=\frac{l(1+e^2+3e^4)}{1+e^2-e^4}$$

1. In any conic, prove that the sum of reciprocals of two perpendicular focal chord is constant

Proof: Let the equation of the conic be

$$\frac{l}{r}=1+e\cos\theta \qquad \text{---(1)}$$



Let PSP', QSQ' be the perpendicular focal chords of the conic (1)

Let the vectorial angle of P be α

Let the vectorial angle of Q be $\frac{\pi}{2}+\alpha$

Let the vectorial angle of P' be $(\pi+\alpha)$

Let the vectorial angle of Q' be $\frac{3\pi}{2}+\alpha$

$$\therefore \qquad \frac{l}{SP}=1+e\cos\alpha$$

$$\frac{l}{SP} = 1 + e\cos\left(\frac{\pi}{2} + \alpha\right) = 1 - e\sin\alpha$$

$$\frac{l}{SP'} = 1 + e\cos(\pi + \alpha) = 1 - e\cos\alpha$$

$$\frac{l}{SQ'} = 1 + e\cos\left(\frac{3\pi}{2} + \alpha\right) = 1 - e\sin\alpha$$

$$\therefore \qquad SP = \frac{l}{1 + e\cos\alpha}$$

$$SP' = \frac{l}{1 - e\cos\alpha}$$

$$SQ = \frac{l}{1 - e\sin\alpha}$$

$$SQ' = \frac{l}{1 - e\sin\alpha}$$

$$PP' = SP + SP' = \frac{l}{1 + e\cos\alpha} + \frac{l}{1 - e\cos\alpha}$$

$$= \frac{l[1 - e\cos\alpha + 1 + e\cos\alpha]}{(1 + e\cos\alpha)(1 - e\cos\alpha)}$$

$$= \frac{2l}{1 - e^2\cos^2\alpha}$$

$$QQ' = SQ + S'Q = \frac{l}{1 - e\sin\alpha} + \frac{l}{1 + e\sin\alpha}$$

$$= \frac{l[1 - e\sin\alpha + 1 + e\sin\alpha]}{(1 - e\sin\alpha)(1 + e\sin\alpha)}$$

$$= \frac{2l}{1 - e^2\sin^2\alpha}$$

$$\therefore \frac{1}{PP'} + \frac{1}{QQ'} = \frac{1 - e^2\cos^2\alpha}{2l} + \frac{1 - e^2\cos^2\alpha}{2l}$$

$$= \frac{1}{2l}\left[2 - e^2\left(\cos^2\alpha + \sin^2\alpha\right)\right]$$

$$= \frac{1}{2l}\left(2 - e^{2}\right) = a \text{ constant}$$

2. Prove that the perpendicular focal chords of a rectangular hyperbola are equal.

Proof:

Asian prob (1), pute = $\sqrt{2}$

$$\frac{1}{PP'} + \frac{1}{QQ'} = \frac{1}{2l}[2-2] = 0$$

$$\therefore \frac{1}{PP'} = -\frac{1}{QQ'}$$

$$\therefore PP' = -QQ'$$

$\therefore PP' = QQ'$ (in magnitude)

3. PSP', QSQ' are two focal chords of a conic cutting each other at right angles.

Prove that $\frac{1}{SP.SP'} + \frac{1}{SQ.SQ'}$ = a constant

Solution : Let PSP' and QSQ' be two focal chords of the conic

$$\frac{l}{r} = 1 + e\cos\theta \qquad \text{---(1)}$$

Let the vectorial angle of P be α

$$\therefore \qquad Q \; is \; \frac{\pi}{2} + \alpha, P' \; is \; \pi + \alpha, Q' \; is \left(\frac{3\pi}{2} + \alpha\right)$$

The points $P(SP, \alpha)$ $\quad Q\left(SQ, \frac{\pi}{2} + \alpha\right)$,

$P'(SP', \pi + \alpha)$ $\quad Q'\left(SQ', \frac{3\pi}{2} + \alpha\right)$ ---(1) lie on

$$\therefore \qquad \frac{l}{SP} = 1 + e\cos\alpha;$$

$$\frac{l}{SQ} = 1 + e\cos\left(\frac{\pi}{2} + \alpha\right) = 1 - e\sin\alpha$$

$$\frac{l}{SP'} = 1 + e\cos(\pi + \alpha) = 1 - e\cos\alpha$$

$$\frac{l}{SQ'} = 1 + e\cos\left(\frac{3\pi}{2} + \alpha\right) = 1 + e\sin\alpha$$

$$\therefore \qquad \frac{l}{SP} = \frac{1 + e\cos\alpha}{l}; \qquad \frac{l}{SP'} = \frac{1 - e\cos\alpha}{l}$$

$$\therefore \qquad \frac{l}{SP}\cdot\frac{l}{SP'} = \frac{(1 + e\cos\alpha)(1 - e\cos\alpha)}{l^2}$$

$$= \frac{1}{l^2}(1 - e^2\cos^2\alpha) \qquad \text{---(2)}$$

$$\frac{l}{SQ}\cdot\frac{l}{SQ'} = \frac{1 - e\sin\alpha}{l} \times \frac{1 + e\sin\alpha}{l}$$

$$= \frac{1}{l^2}\left[(1 - e\sin\alpha)(1 + e\sin\alpha)\right]$$

$$= \frac{1}{l^2}(1 - e^2\sin^2\alpha)$$

(2) + (3) gives

$$\frac{l}{SP}\cdot\frac{l}{SP'} + \frac{l}{SQ}\cdot\frac{l}{SQ'} = \frac{1 - e^2\cos^2\alpha}{l^2} + \frac{1 - e^2\sin^2\alpha}{l^2}$$

$$= \frac{1}{l^2}\left[2 - e^2(\cos^2\alpha + \sin^2\alpha\right]$$

$$= \frac{1}{l^2}(2 - e^2)$$

$$= a \quad \text{constant.}$$

10.7. References

Analytical Geometry by T.K. Manickavasagam Pillai.

Unit IV

Lesson – 11

Analytical Geometry of Three Dimensions

Contents

11.0 Aims and Objectives

In this lesson we concentrate on the concept of the equations of a straight live, sphere, including the fundamental of the planes and straight lives. Also we study the shortest distance between two lines.

Results:

1) The distance between two points A(x_1,y_1,z_1) and B(x_2,y_2,z_2) is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

2) If a straight line makes angles α, β, γ with the coordinate axes, then the direction cosines of the line are defined as $\cos\alpha, \cos\beta, \cos\gamma$. These are denoted by l, m, n.

Note that $l^2 + m^2 + n^2 = 1$.

3) If $\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$, then the direction cosines of $\overrightarrow{OP}$ are $\dfrac{x}{|\overrightarrow{OP}|}, \dfrac{y}{|\overrightarrow{OP}|}, \dfrac{z}{|\overrightarrow{OP}|}$

4) A set of numbers a,b,c which are proportional to he direction cosines *l*,m,n of a line are called the direction ratios of that line.

If a,b,c are direction ratios of a line then the direction cosines of that line are

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}} \quad ,m = \frac{b}{\sqrt{a^2+b^2+c^2}} \quad ,n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

5) The equation of a straight line passing through a point (x_1,y_1,z_1)having direction ratios

l,m,n is $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$

6) Equations of straight line passing through the points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

7) The direction ratios of the straight line joining the points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ are

$x_2-x_1, y_2-y_1, z_2-z_1$

8) The condition for the lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_1}{l_2} = \frac{y-y_1}{m_2} = \frac{z-z_1}{n_2}$ to be

perpendicular is $l_1l_2 + m_1m_2 + n_1n_2 = 0$ and to be parallel is $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$

9) If a point P divides the line joining the points $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$ internally in

the ratio m:n, then P is given by $P\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+yn_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$

10) If the point P divides AB externally where $A(x_1,y_1,z_1)$ and $B(x_2,y_2,z_2)$, then

$$P = \left(\frac{mx_2-nx_1}{m-n}, \frac{my_2-yn_1}{m-n}, \frac{mz_2-nz_1}{m-n}\right)$$

11) Two planes intersect on a straight line. The general equation of a plane is of the form ax+by+cz+d=0. a,b,c are called the direction rations of the normal to this plane.

12) Equation of any plane passing through the point (x_1,y_1,z_1) is $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$.

13) Angle between two planes is defined as angle between their normals.

Let the equations of two planes be

$a_1x+b_1y+c_1z+d_1=0$ and $a_2x+b_2y+c_2z+d_2=0$

a). Then the angle between them is

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

b) The condition for these planes to be parallel is $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

c) The condition for these planes to be perpendicular is $a_1a_2 + b_1b_2 + c_1c_2 = 0$

14) Equation of the plane containing the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = 0$ is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Where a*l*+bm+cn = 0.

15) Consider equations of the plane and the line as ax+by+cz+d=0 ---(1) and

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \quad \text{---(2)}$$

The condition for the plane and line to be parallel is a*l*+bm+cn = 0 and to be perpendicular is $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$

16) The length of perpendicular from the point (x_1,y_1,z_1) to the plane ax+by+cz+d=0 ---(1) is

$$\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

a) The length of perpendicular from the origin to the plane (1) is $\left|\frac{d}{\sqrt{a^2 + b^2 + c^2}}\right|$

b) The distance between the parallel planes ax+by+cz+d=0 and ax+by+cz+d_1=0 is $\left|\frac{d - d_1}{\sqrt{a^2 + b^2 + c^2}}\right|$

11.1 Coplanar lines

1. Symmetrical form of straight line

The equation of the line $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ is called they symmetrical form of a straight line.

2. Since two planes intersect in a straight line, its equation can be put in the form

$$ax+by+cz+d=0 = a_1x+b_1y+c_1z+d_1=0$$

This is called non – symmetrical form of a straight line.

11.2 Examples Model 1:

1. Put in symmetrical form the lines

$$3x-2y+z-1 = 0 = 5x+4y-6z-2 \qquad \text{---(1)}$$

Step 1: Let l,m,n be the direction rations of the given line

$$\therefore 3l-2m+n=0 \qquad \text{---(1)}$$

$$5l+4m-6n=0 \qquad \text{---(2)}$$

By rule of cross multiplication

$$\begin{array}{cccc} & l & m & n \\ -2 & 1 & 3 & -2 \\ 4 & -6 & 5 & 4 \end{array}$$

$$\frac{l}{12-4}=\frac{m}{5+18}=\frac{n}{12+10}$$

$$\frac{l}{8}=\frac{m}{23}=\frac{n}{22}$$

Then the direction cosines of the line are proportional to 8,23,22 (which are nothing but d.r's)

Step 2: To find any point on the line (1).

Let the line (1) meet xy plane

$\therefore$ put z = 0 in (1)

$\therefore$ $3x-2y-1 = 0,$

$5x+4y-2 = 0$

$\therefore$ By rule of cross multiplication

$$\begin{array}{cccc} & x & y & 1 \\ -2 & -1 & 3 & -2 \\ 4 & -2 & 5 & 4 \end{array}$$

$$\frac{x}{4+4}=\frac{y}{-1+6}=\frac{1}{12+10}$$

$$\frac{x}{8}=\frac{y}{5}=\frac{1}{22}$$

$\therefore \quad x=\frac{8}{22}, \qquad y=\frac{5}{22}$

$\therefore$ Any point on the lines $\begin{pmatrix} \frac{8}{22}, & \frac{5}{22}, & 0 \\ x_1, & y_1, & z_1 \end{pmatrix}$

$\therefore$ Equation of the line in symmetrical form is

$$\frac{x-\frac{8}{22}}{8}=\frac{y-\frac{5}{22}}{23}=\frac{z}{-22}$$

2) Prove that the lines

$3x-4y+2z=0=-4x+y+3z$ ----(1) and

$x+3y-5z+9=0=7x-5y-z+7$ ----(2) are parallel

Solution step 1:

Let l,m,n be the direction ratios of the line (1)

$\therefore \quad 3l-4m+2n=0,$

$-4l+m+3n=0$

$\therefore$ By rule of cross multiplication

$$\begin{array}{ccccc} & l & m & n & \\ -4 & 2 & 3 & -4 & \\ 1 & 3 & -4 & 1 & \end{array}$$

$$\frac{l}{-12-2}=\frac{m}{-8-9}=\frac{n}{3-16}$$

$$\frac{l}{-14}=\frac{m}{-17}=\frac{n}{-13} \text{ or } \frac{l}{14}=\frac{m}{17}=\frac{n}{13}$$

$\therefore$ The d.cs' of line (1) are proportional to

+14 , +17, +13

a_1 b_1 c_1

Step 2: Let *l*,m,n be the d.r's of the line (2)

$$\therefore \quad l-3m-5n=0,$$

$$7l-5m-n=0$$

$$\begin{array}{cccc} & l & m & n \\ 3 & -5 & 1 & 3 \\ -5 & -1 & 7 & -5 \end{array}$$

$$\frac{l}{-3-25}=\frac{m}{-35+1}=\frac{n}{-5-21}$$

$$\frac{l}{-28}=\frac{m}{-34}=\frac{n}{-26} \text{ or } \frac{l}{14}=\frac{m}{17}=\frac{n}{13}$$

$\therefore$ The d.cs' of line (2) are proportional to

+14 , +17, +13

a_2 b_2 c_2

$$\therefore \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$$

$\therefore$ The lines (1) and (2) are parallel

3) Prove that the lines

$2x+y+3z-7=0=x-2y+z-5$ and

$4x+4y-8z=0=10x-8y+7z$ are at right angles.

Solution: Proceed as before in example (2), check $a_1a_2+b_1b_2+c_1c_2=0$ for right angles (ie. Perpendicular)

Condition for Two straight lines to be coplanar consider two straight lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \qquad \text{----(1) and}$$

$$\frac{x-x_1}{l_2}=\frac{y-y_1}{m_2}=\frac{z-z_1}{n_2} \qquad \text{----(2)}$$

Equation of the plane through the line (1) is

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0 \qquad \text{----(3)}$$

Where $Al_1 + Bm_1 + Cn_1 = 0$ ----(2)

If the plane (3) contains the line (1), then (x_2,y_2,z_2) lies on the plane (3)

$\therefore$ Put x= x_2, y = y_2 , z = z_2 in (3)

$$A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0 \qquad \text{----(4)}$$

Also the line (2) is perpendicular to the normal to the plane (3)

$$Al_2 + Bm_2 + Cn_2 = 0$$

Where $A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0$

$$Al + Bm + Cn = 0$$

$$Al_1 + Bm_1 + Cn_1 = 0$$

Eliminate A,B,C between the above, we get

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Equation of the plane containing the line (1) and (2) is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Model 1: Two symmetrical form of straight lines are given.

Examples: 1. Show that the lines

$$\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3}; \qquad \frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$$

are coplanar and find the equation of the plane containing them.

Solution : Let $L_1 = \frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3} = r_1$

$$\frac{x+3}{2} = r_1 \quad \Bigg| \quad \frac{y+5}{3} = r_1 \quad \Bigg| \quad \frac{z-7}{-3} = r_1$$

$$x + 3 = 2r_1; \qquad y + 5 = 3r_1; \qquad z - 7 = -3r_1;$$

$$x = -3 + 2r_1 \qquad y = 3r_1 - 5, \qquad z = -3r_1 + 7$$

$\therefore$ Any point on L_1 $is\ (2r_1 - 3, 3r_1 - 5, -3r_1 + 7)$ ----(A)

Let $L_2 : \dfrac{x+1}{4} = \dfrac{y+1}{5} = \dfrac{z+1}{-1} = r_2$

$$\frac{x+1}{4} = r_2 \quad \Bigg| \quad \frac{y+1}{5} = r_2 \quad \Bigg| \quad \frac{z+1}{-1} = r_2$$

$x + 1 = 4r_2;\quad y + 1 = 5r_2;\quad z + 1 = -r_2$

$x = 4r_2 - 1;\quad y = 5r_2 - 1;\quad z = -r_2 - 1$

$\therefore$ Any point on L_2 $is\ (4r_2 - 1, 5r_2 - 1, -r_2 - 1)$ ----(B)

If the lines L_1 and L_2 are coplanar, then A and B represent the same point.

$2r_1 - 3 = 4r_2 - 1;\quad 3r_1 - 5 = 5r_2 - 1;\quad -3r_1 + 7 = -r_2 - 1$

$4r_2 - 2r_1 = -2 \quad ---(1)$

$3r_1 - 5r_2 = 4 \quad ---(2)$

$-3r_1 + r_2 = 8 \quad ---(3)$

Solve (1) and (3)

$r_1 = 3;\quad r_2 = 1$

Using in (3), the equation if statisfied.

$\therefore$ The lines L_1 and L_2 are coplanar

Step 2: To find the point of intersection Put r_2=1 in (B), the point of intersection is (3,4,-2)

Step 3: Equation of the plane containing them is

$$\begin{vmatrix} x+3 & y+5 & z-7 \\ 2 & 3 & -3 \\ & 5 & -1 \end{vmatrix} = 0$$

$(x+3)(-3+15) - (y+5)(-2+12) + (z-7)(10-12) = 0$

$12(x+3) - 10(y+5) - 2(z-7) = 0$

$12x + 36 + 10y - 50 - 2z + 14 = 0$

$12x - 10y - 2z = 0$

$\div 2$;

6x-5y-z=0

Model 2: Given a symmetrical form of a line and a non-symmetrical form of a line.

Prove that the lines $\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}$ and $x+2y+3z-8=0=2x+3y+4z+1$ intersect and find their point of intersection. Find also the equation of the plane containing them.

Solution: $L_1:\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3}$ ---(1)

$L_2:$ $x+2y+3z-8=0=2x+3y+4z-11$ ---(2)

Equation of any plane through the line (2) is

$$x+2y+3z-8+\lambda(2x+3y+4z-11)=0 \qquad \text{---(3)}$$

L_1 passes through (-1, -1, -1)

L_1 lies on the plane (3) if (-1, -1, -1) lies on (3)

If (-1-2-3-8) + λ (-2-3-4-11) = 0

$\Rightarrow \lambda=\frac{-7}{10}$

Using $\lambda=\frac{-7}{10}$ in (3)

$$x+2y+3z-8-\frac{7}{10}(2x+3y+4z-11)=0$$

Simplifying we get the equation of the plane (3) is

4x+y-2z-3=0 ---(4)

The d.r's of the normal to this plane

4, 1, -2

a b c

The d.r's of the line L_1: 1, 2, 3

l m n

$\therefore$ $al+bm+cn=4+2-6=0$

$\therefore$ The line L_1 lies in the plane (4)

$\therefore \quad L_1 \ and \ L_2$ are coplanar

Step 2: To find the point of intersection.

Any point on L$_1$:

Let $\frac{x+1}{1} = \frac{y+1}{2} = \frac{z+1}{3} = r_1$(say)

$\therefore \ x = r_1 - 1, \quad y = 2r_1 - 1; \quad z = 3r_1 - 1;$

$\therefore$ Any point on $L_1 \ is \ (r_1 - 1, 2r_1 - 1, 3r_1 - 1)$ ----(A)

This point lies on the plane x+2y+3z-8=0

Then $L_1 \ is \ r_1 - 1 + 2(2r_1 - 1) + 3(3r_1 - 1) - 8 = 0$

$r_1 - 1 + 4r_1 - 2 + 9r_1 - 3 - 8 = 0$

$14r_1 - 14 = 0$

$r_1 = 1$

Put $r_1 = 1$ in (A), the point of intersection is (o,1,2).

Model 3: Two straight lines are given in non-symmetrical form prove that the straight lines.

$x + y + z - 3 = 0 = 2x + 3y + 4z - 5 = 0$ ---(1) and

$4x - y + 5z - 7 = 0 = 2x - 5y - z - 3$ ---(2) are coplanar. Find the point of intersection and find the equation of the plane containing them.

Solution : Reduce the line (1) to symmetrical form and proved as in model 2.

Ans:

a) They are coplanar
b) Point of intersection is (4, -1, 0)
c) Equation of the plane containing them is x+2y+3z-2-0.

11.3. Shortest distance between two lines.

Definition 1: Two straight lines are called skew if they are neither intersecting nor parallel.

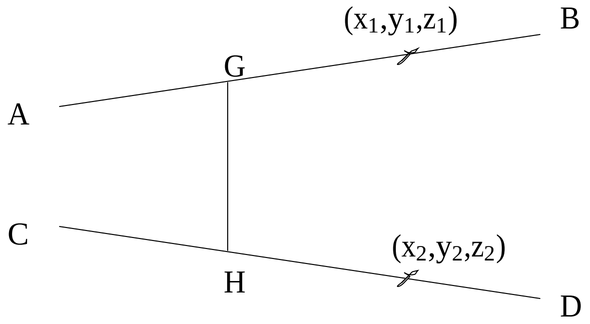
Definition 2: If two straight lines are skew, then there will be one and only are perpendicular common to both the straight lines , then this common perpendicular is known as the shortest distance between the two lines.

11.3.1. Find the short distance between the skew lines and find also find its equation.

$L_1 : \frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and

$L_2 : \frac{x-x_1}{l_2} = \frac{y-y_1}{m_2} = \frac{z-z_1}{n_2}$

Let AB and CD represent the two lines L_1 and L_2 respectively.



Let GH be the shortest distance between the lines L_1 and L_2

L_1 passes through $A(x_1,y_1, z_1)$ and L_2 passes through $B (x_2,y_2,z_2)$

Equation of any plane containing L_1 is

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0 \qquad \text{---(1)}$$

Where $al_1+bm_1+cn_1=0$ ---(2)

The plane (1) is parallel to L_2

$\therefore \quad al_2+bm_2+cn_2=0$ ---(3)

Solve (2) and (3)

$$al_1+bm_1+cn_1=0$$

$$al_2 + bm_2 + cn_2 = 0$$

$$\begin{array}{cccc} a & b & c & \\ m_1 & n_1 & l_1 & m_1 \\ m_2 & n_2 & l_2 & m_2 \end{array}$$

$$\frac{a}{m_1n_2 - m_2n_1} = \frac{b}{m_1l_2 - l_1n_2} = \frac{c}{l_1m_2 - l_2m_1}$$

$\therefore$ Using in (1)

$$(m_1n_2 - m_2n_1)(x - x_1) + (m_1l_2 - l_1n_2)(y - y_1) + (l_1m_2 - l_2m_1)(z - z_1) = 0$$

The SD = Perpendicular from the point B (x_2,y_2,z_2) on the plane (4)

$$= \frac{(m_1n_2 - m_2n_1)(x_2 - x_1) + (n_1l_2 - n_1l_2)(y_2 - y_1) + (l_1m_2 - l_2m_1)(z_2 - z_1)}{\sqrt{\sum (m_1n_2 - m_2n_1)^2}}$$

$$\therefore SD = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{\sum (m_1n_2 - m_2n_1)^2}}$$

Step 2: Equation of the plane containing the line L_1 and line GH is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \qquad \text{---(A)}$$

Where *l*, m, n are the d.r's of GH

Equation of the plane containing the line L_2 and the line GH is

$$\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 \qquad \text{---(B)}$$

Equation of A and B together representative equation of the line GH

11.4.Examples : Model 1:

Find the length of the SD between the lines

$L_1 : \frac{x-3}{-3} = \frac{y-8}{1} = \frac{z-3}{-1}$ *and* $\frac{x+3}{3} = \frac{y+7}{-2} = \frac{z-6}{-4}$ and find its equation.

Solution: Step 1

$$L_1 : \frac{x-3}{-3} = \frac{y-8}{1} = \frac{z-3}{-1};$$

$$L_2 : \frac{x+3}{3} = \frac{y+7}{-2} = \frac{z-6}{-4}$$

Let GH be the SD between L_1 and L_2

Let *l*,m,n be the d.r's of GH

GH perpendicular L_1 $\therefore$ $-3l + m - n = 0$

GH perpendicular L_2 $\therefore$ $3l + 2m - 4n = 0$

By rules of cross multiplication

$$\begin{array}{ccccc} & l & m & n & \\ 1 & -1 & -3 & 1 & \\ & & & & \\ -2 & -4 & 3 & -2 & \end{array}$$

$$\frac{l}{-4-2} = \frac{m}{-3-12} = \frac{n}{6-3}$$

$$\frac{l}{-6} = \frac{m}{-15} = \frac{n}{-3}$$

$$\text{or } \frac{l}{-2} = \frac{m}{-5} = \frac{n}{1}$$

$\therefore$ The d.c's of GH are proportional to -2, -5, 1

l , m, n

The d.c's of GH are $\frac{-2}{\sqrt{30}}, \frac{-5}{\sqrt{30}}, \frac{1}{\sqrt{30}},$

L M N

L1 passes through $\begin{array}{ccc} x_1 & y_1 & z_1 \\ (3, & 8, & 3) \end{array}$

L1 passes through $\begin{array}{ccc} (-3, & -7, & 6) \\ x_2 & y_2 & z_2 \end{array}$

$\therefore$ The $SD = (x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n$ (formula)

$$= \frac{-2}{\sqrt{30}}(-6) - \frac{5}{\sqrt{30}}(-7-8) + \frac{1}{\sqrt{30}}(6-3)$$

$$= \frac{12}{\sqrt{30}} + \frac{75}{\sqrt{30}} + \frac{3}{\sqrt{30}}$$

$$= \frac{90}{\sqrt{30}}$$

$$SD = \frac{3 \times 30}{\sqrt{30}} = 3\sqrt{30}$$

Step 2: To find the equation of GH

The equation of SD is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0 = \begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix} = 0$$

$$\begin{vmatrix} x - 3 & y - 8 & z - 3 \\ -3 & 1 & -1 \\ -2 & -5 & 1 \end{vmatrix} = 0 = \begin{vmatrix} x + 3 & y + 7 & z - 6 \\ 3 & -2 & 4 \\ -2 & -5 & 1 \end{vmatrix}$$

$$4x - 5y + 17z - 23 = 0 = 18x - 11y - 19z + 91$$

Model 2: Find the SD between their lines

$$L_1 : \frac{x-5}{3} = \frac{y-6}{-4} = \frac{z-9}{1}$$

$$L_2 : \quad 2x - 2y + z - 3 = 0 = 2x - y + 2z - 9$$

Solution: Any plane through L_2 is

$$(2x-2y+z-3)+\lambda(2x-y+2z-9)=0$$

$$2x-2y+z-3+2\lambda x-\lambda y+2\lambda z-9\lambda=0$$

$$x(2+2\lambda)-y(2+\lambda)+z(1+2\lambda)-3-9\lambda=0 \qquad \text{---(1)}$$

The d.r's of the normal to this plane are $2+2\lambda, -(2+\lambda), 1+2\lambda$

Plane (1) is parallel to L_1

$$\therefore \quad 3(2+2\lambda)+4(2+\lambda)+1+2\lambda=0$$

$$6+6\lambda+8+4\lambda+1+2\lambda=0$$

$$12\lambda+15=0$$

$$12\lambda=-15$$

$$\lambda=\frac{-5}{4}$$

Using (1), the equation of the plane is 2x+3y+6z-33=0

L_1 passes through (5, 6, 9)

$\therefore$ SD = Perpendicular distance from (5, 6, 9) to the plane 2x+3y+6z-33=0

$$=\frac{2(5)+3(6)+6(9)-33}{\sqrt{2^2+3^2+6^2}}$$

$$=\frac{10+18+54-33}{\sqrt{49}}$$

$$=\frac{49}{7}=7\text{ unit.}$$

Model 3:

Find the SD between the lines

$L_1: 3x-9y+5z=0=x+y-z$ and

$L_2: 6x+8y+3z-10=0=x+2y+z-3$

Solution: Reduce L_1 to symmetric form and proceed as proceeded in model 2

Ans: $SD = \frac{13}{\sqrt{342}}$

11.5 Let us sum up

So far we have studied how to prove two lines are coplanar, how to find the shortest distance between two skew lines.

11.6. Check your progress

(1) Find the condition of the lines

$\frac{x-a}{a^1} = \frac{x-b}{{}^1b} = \frac{x-c}{{}^1c}\ and\ \frac{x-a^1}{a} = \frac{y-b^1}{b} = \frac{z-c^1}{c}$ to be perpendicular

(2) Find the direction ratios of the normal to the plane x-y+2-1=0

11.7. Lesson End Activities

1. Show that the lines

$L_1 : \frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}; \qquad L_2 : \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ are coplanar. Find their point of intersection and the equation of the plane containing them

Ans: x-2y+z = 0

2. Show that the lines

$\frac{x-a}{a'} = \frac{y-b}{b'} = \frac{z-c}{c'}\ and\ \frac{x-a'}{a} = \frac{y-b'}{b} = \frac{z-c'}{c}$ are coplanar.

3. Show that the lines

$\frac{x-1}{2} = \frac{y-z}{-1} = \frac{z+b}{-3}\ and$

$x + 2y + z + 2 = 0 = 4x + 5y + 3z + 6$ are coplanar. Prove also that their point of intersection is $\left(\frac{5}{3}, \frac{5}{3}, -7\right)$ and the equation of the plane containing them is 2x+y+z+2=0/

11.8. Points for discussion

1. Prove that the lines

$$\frac{x+1}{1}=\frac{y+z}{2}=\frac{z-b}{1}\ and$$

$x-2y+2z=3, \quad x-4y+5z=8$ are coplanar.

2. Find the length and equation of the SD between the lines

$$\frac{x+3}{-4}=\frac{y-6}{6}=\frac{z}{2}; \qquad \frac{x+2}{-4}=\frac{y}{1}=\frac{z-7}{1}$$

Ans: $SD=\frac{14}{\sqrt{3}}$

Equation : 16x+11y-z-18=0=2x+7y+z-3

11.9. References

Analytical Geometry of Three
Dimensions by N.P Bali

Lesson-12

Contents

12.0 Aim and Objectives

In this lesson we are going to learn in detail about sphere.

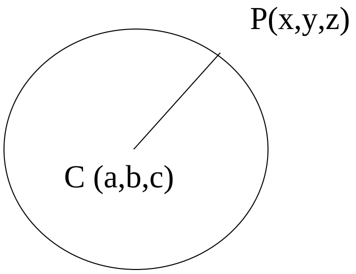
12.1 Sphere

The locus of a variable point in space whose distance from a fixed point is a constant is called a sphere. The fixed point is called its centre and fixed distance is called its radius.

12.2 Examples

1. Find the equation of a circle whose centre is at the point (a,b,c) and radius r units.

Solution :



Let C (a,b,c) be the centre. Let P(x,y,z) be any point on the sphere

Given CP = r

$$\therefore \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}=r$$

Squaring both sides

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Cor: If the centre is at the origin (0,0,0) then equation of the sphere is $x^2+y^2+z^2=r^2$

2. Prove that the equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ represents a sphere. Hence find its centre and radius.

Solution: The given equation is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

adding $u^2+v^2+w^2$ to both sides.

$$(x^2+2ux+u^2)+(y^2+2vy+v^2)+(z^2+2wz+w^2)+d = u^2+v^2+w^2$$

$$(x+u)^2+(y+v)^2+(z+w)^2=u^2+v^2+w^2-d$$

$$[x-(-u)]^2+[y-(-v)]^2+[z-(-w)]^2=\left(\sqrt{u^2+v^2+w^2-d}\right)^2$$

This is of the form $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

Which is a sphere.

$\therefore$ The given equation represents sphere whose centre is at (-u, -v, -w) and

$r=\sqrt{u^2+v^2+w^2-d}$.

Problems 1: Find the radius and the centre of the sphere

$x^2+y^2+z^2-6x-2y-4z-11=0$

Solution: The given equation of the form $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$$\therefore \begin{array}{c|c|c|c} 2u=-6 & 2v=-2 & 2w=-4 & \\ u=-3 & v=-1 & w=-2 & \end{array} \quad d=-11$$

$\therefore$ Centre = (-u, -v, -w) = (3,1,2)

$$r=\sqrt{u^2+v^2+w^2-d}$$

$$=\sqrt{9+1+4+11}$$

$$=\sqrt{25}$$

$r=5$ unit.

2. Find the centre of and radius of the sphere $2x^2+2y^2+2z^2-2x+4y-6z+5=0$

Solution: The given equation of the sphere is

$$2x^2+2y^2+2z^2-2x+4y-6z+5=0$$

$\div$ 2;

$$x^2+y^2+z^2-x+2y-3z+\frac{5}{2}=0$$

$2u=-1;$ $2v=2;$ $2w=-3;$ $d=\frac{5}{2}$

$u=\frac{-1}{2};$ $v=1;$ $w=\frac{-3}{2}$

$\therefore$ Centre = (-u, -v, -w)

$$=\left(\frac{1}{2},-1,\frac{3}{2}\right)$$

$$r=\sqrt{u^2+v^2+w^2-d}$$

$$=\sqrt{\frac{1}{4}+1+\frac{9}{4}-\frac{5}{2}}$$

$$=\sqrt{\frac{5}{2}+1-\frac{5}{2}}$$

$r=1$ unit.

Model 2: Equation of the sphere passing through four given points.

1. Find the equation of the sphere passing through the points (2,3,1), (5,-1,2) (4,3,-1) and (2,5,3)

Solution: Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \qquad \text{---(1)}$$

(1) Passes through the point (2,3,1)

4+9+1+4u+6v+2w+d=0

4u+6v+2w+d=-14 ---(2)

(1) Passes through the point (5,-1,2)

$25+1+4+10u-2v+4w+d=0$

$10u-2v+4w+d=-30$ ---(3)

(1) Passes through the point (4,3,-1)

$16+9+1+8u+6v-2w+d=0$

$8u+6v-2w+d=-26$ ---(4)

(1) Passes through the point (2,5,3)

$4+25+9+4u+10v+6w+d=0$

$4u+10v+6w+d=-38$ ---(5)

Step (2) – (3) gives

$$\begin{array}{r} 4u + 6v + 2w + d = -14 \\ 10u - 2v + 4w + d = -30 \\ \hline \end{array}$$

Sub $-6u + 8v - 2w = 16$

÷ 2;

$-3u + 4v - w = 8$ ---(6)

(3) – (4) gives

$$\begin{array}{r} 10u - 2v + 4w + d = -30 \\ 8u + 6v - 2w + d = -26 \\ \hline \end{array}$$

Sub $2u - 8v + 6w = -4$

÷ 2;

$u - 4v + 3w = -2$ ---(7)

(4) – (5) gives

$$\begin{array}{r} 8u + 6v - 2w + d = -26 \\ 4u + 10v + 6w + d = -38 \\ \hline \end{array}$$

Sub $4u - 4v - 8w = 8$

÷ 4;

or $u - v - 2w = 2$ ---(8)

Step 3: Solve (6). (7), (8) by cramers' solution (or) by solving

$-3u + 4v - w = 8$ ---(6)

$u - 4v + 3w = -2$ ---(7)

$u - v - 2w = 2$ ---(8)

$-3u + 4v - w = 8$

$u - 4v + 3w = -2$

Add $-2u + 2w = 6$

$-u + w = 3$ ---(9)

(7) is $u - 4v + 3w = -2$

(8) x 4 ; $4u - 4v - 8w = 8$

Sub $-3u + 11w = -10$ ---(10)

Solve (9) and (10)

3 x (9) $-3u + 3w = -9$

$-3u + 11w = -10$

Sub $-8w = 19$

$$w = \frac{-19}{8}$$

Using in (9)

$$-u - \frac{19}{8} = 3$$

$$-u = \frac{19}{8} - 3 = -\frac{5}{8}$$

$$u = \frac{5}{8}$$

Put $u = \frac{5}{8}$, $w = \frac{-19}{8}$ in (8) , $v = \frac{27}{8}$

Substitute the values of u, v, w in (4)

d = - 56

using in (1)

$$x^2 + y^2 + z^2 + \frac{10}{8}x + \frac{54}{8}y + \frac{38}{8}z - 56 = 0$$

$$8(x^2 + y^2 + z^2) + 10x + 54y + 38z - 448 = 0$$

Which is the required sphere.

2. Find the equation of the sphere passing through the points (1,3,4), (1,-5,2), (1, -3, 0) and having its centre on the plane x + y + z = 0.

Solution: Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad ---(1)$$

(1) Passes through the point (1,3,4)

2u+6v+8w+d=-20 ---(2)

(1) Passes through the point (1,-5,2)

2u-10v+4w+d=-30 ---(3)

(1) Passes through the point (1,-3,0)

2u+6v+d=-10 ---(4)

The centre (-u, -v, -w) lies on the plane

x + y + z = 0.

$\therefore \quad -u - v - w = 0 \qquad \therefore \quad u + v + w = 0$ ---(5)

Solve (2), (3), (4), (5)

u = -1, v = 3, w = -2, d = 10

we get the equation of the sphere is

$$x^2 + y^2 + z^2 + 2x + 6y + 4z + 10 = 0.$$

3. Find the equation of the sphere passing through the points (1,1,-2), (-1,1,2) and having its centre lies on the line

x + y – z – 1 = 0 = 2x – y + z – 2

Proof: Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \text{---(1)}$$

(1) Passes through the point (1,1,-2)

$$2u+2v-4w+d=-6 \qquad \text{---(2)}$$

(1) Passes through the point (-1,1,2)

$$-2u+2v+4w+d=-6 \qquad \text{---(3)}$$

The centre (-u, -v, -w) lies on the plane

x + y - z -1= 0 = 2x – y + z - 2

$$\therefore \quad -u - v + w - 1 = 0 \qquad -2u + v - w - 2 = 0$$

$$\therefore \quad u + v - w = 1 \qquad or \qquad 2u - v + w + 2 = 0 \qquad \text{---(5)}$$

Solve (2), (3), (4), (5)

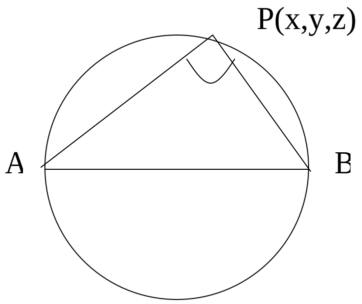
$$u = -1; \qquad v = -\frac{1}{2};\ w = -\frac{1}{2}; d = -5$$

Using in (1) Equation of the sphere is

$$x^2 + y^2 + z^2 - 2x - y - z - 5 = 0 .$$

Equation of a sphere on the line joining the points (x_1,y_1,z_1) as diameter.

Let the points be A (x_1,y_1,z_1) and B(x_2,y_2,z_2)



Let P(x,y,z) be any point on the sphere join AP,PB, APB = 90^0

The d.r's of AP are $x - x_1, y - y_1, z - z_1$

The d.r's of BP $x - x_2, y - y_2, z - z_2$

$\therefore$ AP perpendicular PB

$(x-x_1),(x-x_2)+(y-y_2)(y-y_2)+(z-z_1)(z-z_2)=0$

Which is the required equation of the sphere.

Problems (1) Find the equation of the sphere described on the line joining the points A(4,6,8) and B(-1,3,7) as diameter

Proof: Equation of the sphere is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

$$(x_1,y_1,z_1)=(4,6,8)$$

$$(x_2,y_2,z_2)=(-1,3,7)$$

$\therefore (x-4)(x+1)+(y-6)(y-3)+(z-8)(z-7)=0$

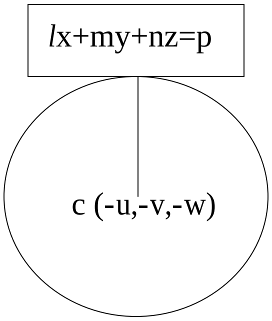
$$x^2+y^2+z^2-3x-9y-15z+70=0$$

12.3 Tangent plane

The locus of all tangent lines drawn to a sphere at a point is called the tangent plane.

12.4.Examples: Find the condition for the plane *l*x+my+nz = p to be tangent plane to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Proof:



The radius of the sphere $\sqrt{u^2+v^2+w^2-d}=r$

P = The perpendicular from the centre of the sphere to the plane *l*x+my+nz = p

$$=\pm\frac{(-lu-mv-nw-p)}{\sqrt{l^2+m^2+n^2}}$$

$$= \pm \frac{(lu + mv + nw + p)}{\sqrt{l^2 + m^2 + n^2}}$$

If the plane lx+my+nz = p is a tangent plane to the sphere,

Then r = p

$$\sqrt{u^2 + v^2 + w^2 - d} = \pm \frac{(lu + mv + nw + p)}{\sqrt{l^2 + m^2 + n^2}}$$

$$\sqrt{u^2 + v^2 + w^2 - d}\sqrt{l^2 + m^2 + n^2} = \pm(lu + mv + nw + p)$$

Squaring both sides

$(u^2+v^2+w^2-d)\ (l^2+m^2+n^2) = (lu+mv+nw+p)^2$

Which is required condition.

Find the equation of the tangent plane to the sphere

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ at the point (x_1,y_1,z_1)

Solution: Equation of the sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \text{---(1)}$$

Equation of any line through (x_1,y_1,z_1) is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \text{ (say)} \qquad \text{---(2)}$$

Any point on the line (2) is $(x_1 + lr,\ y_1 + mr,\ z_1 + nr)$

$$\left\{ \begin{matrix} x - x_1 = lr; & y - y_1 = mr; & z - z_1 = nr - 1; \\ x = x_1 + lr & y = y_1 + mr; & z = z_1 + nr \end{matrix} \right\}$$

$\therefore$ If this point lies on the sphere (1) then

$$(x_1 + lr)^2 + (y_1 + mr)^2 + (z_1 + nr)^2 2 + 2u(x_1 + lr) + 2v(y_1 + mr) + 2w(z_1 + nr) + d = 0$$

$$r^2(l^2 + m^2 + n^2) + 2r\left[lx_1 + my_1 + lu + mv + nw\right] + x_1^{\ 2} + y_1^{\ 2} + z_1^{\ 2} + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$$

$\therefore (x_1, y_1, z_1)$ lies on the sphere

$$x_1^{\ 2} + y_1^{\ 2} + z_1^{\ 2} + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$$

$$\therefore r^2(l^2 + m^2 + n^2) + 2r\left[l(x_1 + n) + m(y_1 + r) + n(z_1 l + w)\right] = 0$$

This is a quadratic in ‘r’, since the line is a tangent line to the sphere then

$$(x-x_1)(x_1+u)+(y-y_1)(y_1+v)+(z-z_1)(z_1+w)=0$$

$$x(x_1+n)-x_1(x_1+n)+y(y_1+v)-y_1(y_1+v)+z(z_1+w)-z_1(z_1+w)=0$$

$$xx_1+ux-x_1^{\ 2}-ux_1+y_1y+vy-y_1^{\ 2}-vy_1+zz_1+wz-z_1^{\ 2}-z_1w=0$$

$$xx_1+yy_1+zz_1+ux+vy+wz=x_1^{\ 2}+y_1^{\ 2}+z_1^{\ 2}+ux_1+vy_1+wz_1=0$$

Add $ux_1+vy_1+wz_1+d$ to both sides

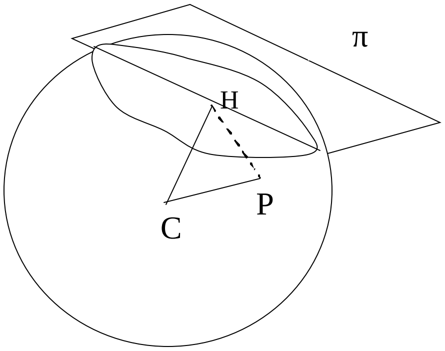
$$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$$

Cor: Equation of the tangent plane at (x_1, y_1, z_1) to the sphere $x^2+y^2+z^2 = a^2$ is

$$xx_1+yy_1+zz_1=a^2$$

Plane section of a sphere

Prove that the plane section of a sphere is a circle.



Let the centre of the sphere be C. Whose radius is r units

Let π be the plane.

Draw CN perpendicular to the plane π

Let P be any point on the section of the sphere by a plane

Then OP = radius of the sphere

In $\Delta CNP \qquad CNP=90^0$

$$\therefore \qquad CP^2=CN^2-NP^2$$

$$NP^2=CP^2-CN^2$$

$$=r^2-p^2 \qquad \text{where CN = P}$$

$$\therefore\ NP = \sqrt{r^2 - p^2} = a \quad \text{constant}$$

$\therefore$ The focus of P is a circle and N is the centre of the circle.

$\therefore$ The plane section of a sphere is a circle. Such a circle is called a small circle.

Note:

1) If S=0, P=0 represent the equation of a sphere and the plane respectively, then the equation of the circle is S=0, P=0
2) If the plane π passes through the centre of the sphere, then the circle is called a great circle.
3) Equation of any sphere passing through the circle S=0, P=0 is
   $S + \lambda P = 0$ where λ is a constant to be determined.

1) Find the centre and radius of the circle

$$x^2 + y^2 + z^2 - 2y - 4z - 11 = 0, x + 2y + 2z = 15$$

Proof: The equation of the sphere is $x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$

This is of the form

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

$2u = 0,\qquad 2v = -2;\qquad 2w = -4;\qquad d = -11$

$u = 0,\qquad v = -1,\qquad w = -2,\qquad d = -11$

$\therefore$ Centre = C(0,1,2)

$r = \sqrt{u^2 + v^2 + w^2 - d}$

$= \sqrt{0 + 1 + 4 + 11} = \sqrt{16} = 4$

$r = 4$

p = The length of perpendicular distance from the centre (0,1,2) of the sphere to the plane x+2y+2z-15=0,

$$= \pm\frac{(0 + 2 + 4 - 15)}{\sqrt{1 + 4 + 4}} = \pm\frac{(-9)}{\sqrt{9}}$$

$$= \pm\frac{(-9)}{3}$$

p=3

$\therefore$ Radius of the circle

$$= \sqrt{r^2 - p^2}$$

$$= \sqrt{4^2 - 3^2}$$

$$= \sqrt{16 - 9}$$

$$= \sqrt{7}$$

Step 2: To find the centre of the circle.

The equation of the plane is

x+2y+2z=15

The d.r's of the normal are 1,2,2

l , m, n

$\therefore$ The d.r's of CN are 1, 2, 2

The line CN passes through (0, 1, 2)

x_1, y_1, z_1

$\therefore$ Equation of CN is $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$

$$\therefore \qquad \frac{x - 0}{1} = \frac{y - 1}{2} = \frac{z - 2}{2} = R \text{ (say)}$$

Any point on this line is (R, 2R+1, 2R+2) ---(A)

Let this be the point N.

If this point lies on the plane

x+2y+2z=15, then

R+2(2R+1)+2(2R+2)-15=0

R+4R+2+4R+4-15=0

9R-9=0

9R=9

R=1

Put R = 1 in (A)

The centre if the circle is (1, 3, 4)

2. Find the equation of the sphere for which the circle

$$x^2 + y^2 + z^2 + 7y - 2z + 2 = 0,\ 2x + 3y + 4z = 8$$

Solution : $S = x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$

$P = 2x + 3y + 4z - 8 = 0$

Equation of any sphere through the circle

S=0, P = 0 is $S + \lambda P = 0$

$$x^2 + y^2 + z^2 + 7y - 2z + 2 + \lambda(2x + 3y + 4z - 8) = 0 \qquad \text{---(1)}$$

$$x^2 + y^2 + z^2 + 2\lambda x + y(7 + 3\lambda) + z(4\lambda - 2) + 2 - 8\lambda = 0$$

Centre = $\left(-\lambda, \frac{-(7+3\lambda)}{2}, \frac{-(4\lambda - 2)}{2}\right)$

This lies on the plane 2x+3y+4z-8=0

$\therefore$ Put $n = -\lambda;\ y = -\frac{1}{2}(7 + 3\lambda) : z = -\frac{1}{2}(4\lambda - 2)$

$\therefore \qquad -2\lambda - \frac{3}{2}(7 + 3\lambda) - 4.\frac{1}{2}(4\lambda - 2) - 8 = 0$

$-4\lambda - 3(7 + 3\lambda) - 4.(4\lambda - 2) - 16 = 0$

$-4\lambda - 21 - 9\lambda - 16\lambda + 8 - 16 = 0$

$-29\lambda - 29 = 0$

$\therefore \quad \lambda = -1$

Put $\lambda = -1$ in (1)

$$x^2 + y^2 + z^2 + 7y - 2z + 2 - 1(2x + 3y + 4z - 8) = 0$$

$$x^2 + y^2 + z^2 + 7y - 2z + 2 - 2x - 3y - 4z + 8 = 0$$

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$$

3. Prove that the circles

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0, \quad 5y + 6z + 1 = 0$

$x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0, \quad x + 2y - 7z = 0$

lie on the same sphere. Find its equation?

Step 1: Equation of the sphere through the first circle

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 + \lambda(5y + 6z + 1) = 0$

$x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 + 5\lambda y + 6\lambda z + \lambda = 0$

$$x^2+y^2+z^2-2x+y(3+5\lambda)+z(4+6\lambda)-5+\lambda=0 \qquad \text{---(1)}$$

Equation of the sphere through the second circle is

$$x^2+y^2+z^2-3x-4y+5z-6+\mu(x+2y-7z)=0$$

$$x^2+y^2+z^2-3x-4y+5z-6+\mu x+2\mu y-7\mu z=0$$

$$x^2+y^2+z^2+x(\mu-3)+y(2\mu-4)+z(5-7\mu)-6=0 \qquad \text{---(2)}$$

The two circles lie on the same sphere if (1) and (2) represent the same sphere

$$\therefore \underset{(i)}{\frac{-2}{\mu-3}} = \underset{(ii)}{\frac{3+5\lambda}{2\mu-4}} = \underset{(iii)}{\frac{4+6\lambda}{5-7\mu}} = \underset{(iv)}{\frac{-5+\lambda}{-6}}$$

From (i) and (ii),

$$-4\mu+8=(\mu-3)(3+5\lambda)$$

$$-4\mu+8=3\mu+5\lambda\mu-9-15\lambda$$

$$\therefore\ -15\lambda+7\mu+5\lambda\mu-17=0 \qquad \text{---(3)}$$

From (i) and (ii)

$$\frac{-2}{\mu-3}=\frac{-5+\lambda}{-6}$$

$$12=(-5+\lambda)(\mu-3)$$

$$12=-5\mu+15+\lambda\mu-3\lambda$$

$$\therefore\ 3\lambda+5\mu-\lambda\mu-3=0 \qquad \text{---(4)}$$

$$-15\lambda+7\mu+5\lambda\mu-17=0$$

(4) x 5: $15\lambda+25\mu-5\lambda\mu-15=0$

Add $\overline{32\mu-32}=0$

$$32\mu=32$$

$$\mu=1$$

Put $\mu=1$ in (i) and (ii)

$$\frac{-2}{-2}=\frac{3+5\lambda}{2-4}$$

$$-2=3+5\lambda$$

$5\lambda = -5$

$\lambda = -1$

Using (iii) and (iv)

$$\frac{4+6\lambda}{5-7\mu} = \frac{4-6}{5-7} = \frac{-2}{-2} = 1$$

$\therefore$ The two circles lie on the same sphere

To find its equation

Step 2: Put $\mu = 1$ in (2)

$x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$

4. Find the equation of the tangent plane to the sphere

$x^2 + y^2 + z^2 + 6x - 2y - 4z = 35$ at the point (3,4,4)

Solution:

$$\begin{array}{l|l|l|} 2x = 6 & 2v = -2 & 2w = -4 \\ u = 3 & v = -1 & w = -2 \end{array}$$

Equation of the tangent at (x_1,y_1,z_1) is

$xx_1 + yy_1 + zz_1 + 3x + 3x_1 - y - y_1 - 2z - 2z_1 - 35 = 0$

$\therefore$ Equation of the tangent plane at (3,4,4) is

$3x + 4y + 4z + 3x + 9 - y - 4 - 2z - 8 - 35 = 0$

$6x + 3y + 2z - 28 = 0$

5. Find the tangent planes to the sphere $x^2 + y^2 + z^2 - 4x - 2y - 6z + 5 = 0$ which are parallel to the plane x+4y+8z=0 and find the point of contact.

Solution: Equation of any plane parallel to the plane x+4y+8z=0 is x+4y+8z+R=0 ---(1)

The centre of the sphere is (2,1,3)

$$r = \sqrt{4+1+9-5} = \sqrt{9} = 3$$

$\therefore$ P = the length perpendicular from (2,1,3) on the plane x+4y+8z+R=0

$$= \left| \frac{2+4+24+R}{\sqrt{1+16+64}} \right|$$

$$= \pm \frac{(30+R)}{9} = 3\,(\text{radius})$$

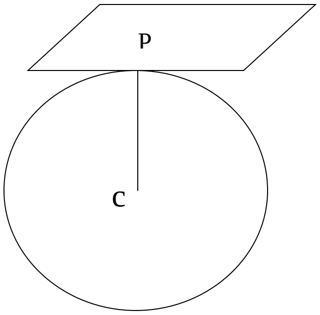
$$\begin{array}{l|l}
= \pm(30+R) = 27 & \quad -(30+R) = 27 \\
30 + R = 27 & \quad 30 + R = -27 \\
R = -3 & \quad R = -57
\end{array}$$

Using in (1) the equations of tangent planes are $x+4y+8z-3=0$ and

$x+4y+8z-57=0$

Step 2: To find the point of contact

Let P be the point of contact of the plane and the sphere



The d.r's of CP are 1,4,8

The line CP passes through C(2,1,3)

Equation of CP is $\frac{x-2}{1} = \frac{y-1}{4} = \frac{z-3}{8} = R\,(\text{say})$

Any point on CP is (R+2, 4R+1, 8R+3) ---(A)

If this lies on the plane x+4y+8z=57

$\therefore$ R+2+4(4R+1)+8(8R+3)-57=0

R+2+16R+4+64R+24-57=0

8R-27=0

8R = 27

$$R = \frac{27}{81} = \frac{1}{3}$$

Put $R = \frac{1}{3}$ in (A), the point of conduct is $\left(\frac{1}{3}+2,\ \ \frac{4}{3}+1,\ \ \frac{8}{3}+3\right)$

$$=\left(\frac{7}{3}, \ \frac{7}{3}, \ \frac{17}{3}\right)$$

Step 3: The other tangent plane is x+4y+8z-3=0

To find the point of contact.

If the point (A) lies on this plane

x+4y+8z-3=0, then

R+2+4(4R+1)+8(8R+3)-3=0

R+2+16R+4+64R+24-3=0

81R+27=0

81R = -27

$$R == \frac{+1}{3}$$

Put $R = -\frac{1}{3}$ in (A), the point of conduct is $\left(\frac{5}{3}, \ \frac{-1}{3}, \ \frac{1}{3}\right)$

6. Find the equation of the spheres which pass through the circle $x^2+y^2+z^2=5$, x+2y+3z=3 and touch the plane 4x+3y=15.

Solution: Equation of any sphere through the given circle is

$$x^2+y^2+z^2-5+\lambda(x+2y+3z-3)=0 \qquad \text{---(1)}$$

$$x^2+y^2+z^2+\lambda x+2\lambda y+3\lambda z-3\lambda -5=0$$

$$\text{Centre} = \left(\frac{-\lambda}{2}, -\lambda, \frac{-3\lambda}{2}\right)$$

$$r = \sqrt{\frac{\lambda^2}{4}+\lambda^2+\frac{9\lambda^2}{4}+3\lambda+5}$$

$$= \sqrt{\frac{14\lambda^2}{4}+3\lambda+5}$$

$$= \sqrt{\frac{14\lambda^2+12\lambda+20}{4}}$$

P = Perpendicular from $\left(\frac{-\lambda}{2}, -\lambda, \frac{-3\lambda}{2}\right)$ to the plane 4x+3y-15=0

$$= \pm \frac{\left(-4\times\frac{\lambda}{2} - 3\lambda - 15\right)}{\sqrt{16+9}}$$

$$= \pm \frac{(-2\lambda - 3\lambda - 15)}{5}$$

$$= \pm \frac{(-5\lambda - 15)}{5}$$

$r = p$

$$\sqrt{\frac{14\lambda^2 + 12\lambda + 20}{4}} = \pm(\lambda + 3)$$

$$14\lambda^2 + 12\lambda + 20 = 4(\lambda + 3)^2$$

On simplification $\lambda = \frac{2}{1} - \frac{4}{5}$

Using in (1) the equations of the spheres are $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0$ and

$5x^2 + 5y^2 + 5z^2 - 4x - 8y - 12z - 13 = 0$

12.5 Let us sum up

So far we have studied and learnt about equations of a sphere in different forms, the equation of the tangent plane to a sphere and in finding the radium centre of the circle which is the intersection of the plane and the sphere.

12.6. Check your progress

(1) Find the centre of the sphere

$x^2+y^2+i^2-2x+4y-6z-11=0$

(2) Find the equation of the sphere having (1,2,3) and (3,2,1) as diameter.

12.7. Lesson End Activities

1. Find the center and radius of the sphere $3x^2 + 3y^2 + 3z^2 + 12x - 8y - 10z + 10 = 0$

$$r = \sqrt{\frac{47}{3}}; \quad C = \left(-2, \frac{4}{3}, \frac{5}{3}\right)$$

2. Find the equation of the sphere through the points (1,1,1) (1,2,1), (1,1,2) and (2,1,1)

Ans: $x^2 + y^2 + z^2 - 3x - 3y - 3z + 6 = 0$

3. Find the equation of the sphere which passes through (0,7,7),(1,8,11), (-3,10,7) and centre lies on the plane 2x+2y-z-7=0

Ans: $x^2 + y^2 + z^2 + 2x - 18y - 18z + 154 = 0$

4. Find the equation of the sphere passing through the points (2,1,1) and (0,3,2) and centre lies on the line $2x + y + 3z = 0 = x + 2y + 2z$

Ans: $9(x^2 + y^2 + z^2) + 28x + 7y - 21z - 96 = 0$

5. Find the equation of the sphere through the points (0,0,0) (a,0,0) (0,b,0) and (0,0,c)

Ans: $x^2 + y^2 + z^2 - ax - by - cz = 0$

6. Find the centre and radius of the circle

$x^2 + y^2 + z^2 - 2x - 4y - 11 = 0; \; x + 2y + 2z - 14 = 0$

Ans: $r = \sqrt{7}; \qquad C = (2,4,2)$

7. Find the equation of the sphere in which the circle

$x^2 + y^2 + z^2 - 6x + 3y - z - 8 = 0, 2x + 3y - z + 6 = 0$ is a great circle

Ans: $x^2 + y^2 + z^2 - 4x + 6y + 2z - 2 = 0$

12.8. Points for discussion.

1. Show that the two circles $x^2 + y^2 + z^2 - y + 2z = 0, \; x - y + z - 2 = 0$ and

$x^2 + y^2 + z^2 + x - 3y + z - 5 = 0, \; 2x - y + 4z - 1 = 0$ lie on the same plane.

2. Find the equations of the spheres which pass through the circle

$x^2 + y^2 + z^2 - 4x - y + 3z + 12 = 0, \; 2x + 3y - 7z = 0$ and find the plane x-2y+2z=1

Ans: $x^2 + y^2 + z^2 - 2x + 2y - 4z = 2$

$x^2 + y^2 + z^2 - 6x + 4y - 10z = 22$

3. Find the condition for the plane *l*x+my+nz=P to be a tangent plane to

$x^2 + y^2 + z^2 = a^2$

4. Find the equation of the tangent plane at(x_1,y_1,z_1) on the sphere $x^2 + y^2 + z^2 = a^2$

12.9. References

1. Analytical Geometry of Three Dimensions by N.P. Bali.

Unit V

Lesson - 13

Contents

13.0 Aims and Objectives

Our Aim is to learn lesson, the concept of a cone, right, circular cone, cylinder and conicoids.

13.1. Cone

Definition:

A cone is a surface generated by a line through a fixed point, (the fixed point is called the vertex of the cone) which satisfies one more condition is intersecting a given curve or founding a given surface.

The given curve is called the base curve or the guiding curve and the variable line is called a generator of the cone.

13.2. Examples

Book work 1: Find the equation of a cone with a given vertex and given base

Solution : Let the vertex of the cone be (α, β, γ).

Let the equation of the base be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad z = 0 \qquad \text{---(1)}$$

Any generator through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-0}{n} \qquad \text{---(2)}$$

This line meets the plane z = 0

$\therefore$ Put z = 0 in (2)

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$$

$$\frac{x-\alpha}{l} = \frac{-\gamma}{n}; \quad \frac{y-\beta}{m} = \frac{-\gamma}{n}$$

$$x-\alpha = \frac{-l}{n}\gamma; \quad y-\beta = \frac{-m}{n}\gamma$$

$$x = \alpha \frac{-l}{n}\gamma; \quad y = \beta \frac{-m}{n}\gamma$$

Using in (1)

$$\left(\alpha - \frac{l}{n}r\right)^2 + 2h\left(\alpha - \frac{l}{n}r\right)\left(\beta - \frac{m}{n}r\right) + b\left(\beta - \frac{m}{n}r\right)^2 + 2g\left(\alpha - \frac{l}{n}r\right) + 2f\left(\beta - \frac{m}{n}r\right) + c = 0$$

Eliminate *l*,m,n between (2) and (3)

$$a\left(\alpha - \frac{x-\alpha}{z-\gamma}r\right)^2 + 2h\left(\alpha - \frac{x-\alpha}{z-\gamma}r\right)\left(\beta - \frac{y-\beta}{z-\gamma}r\right) + b\left(\beta - \frac{y-\beta}{z-\gamma}r\right)^2$$
$$+ 2g\left(\alpha - \frac{x-\alpha}{z-\gamma}r\right) + 2f\left(\beta - \frac{y-\beta}{z-\gamma}r\right) + c = 0$$

$$\frac{(\alpha z - x\gamma)^2}{(z-r)^2} + 2h\frac{(\alpha z - x\gamma)(\beta z - y\gamma)}{(z-r)^2} + b\frac{(\beta z - y\gamma)^2}{(z-r)^2} + 2g\frac{(\alpha z - x\gamma)}{(z-r)} + 2f\frac{(\beta z - y\gamma)}{(z-r)} + c = 0$$

Multiply both sides by $(z-r)^2$

$$(\alpha z - x\gamma)^2 + 2h(\alpha z - x\gamma)(\beta z - y\gamma) + b(\beta z - y\gamma)^2 + 2g(\alpha z - x\gamma)(z-r) + 2f(\beta z - y\gamma)(z-r) + c(z-r)^2 = 0$$

Which is the required equation of the cone.

2) Prove that the equation of a cone with its vertex at the origin is a homogeneous equation

Proof: Consider the equation

$$ax^2 + 2hxy + by^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \qquad \text{---(1)}$$

Let P (x_1,y_1,z_1) be any point, on the cone

$\therefore$ Equation of OP is

$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}$ where (1) = (0,0,0)

$\therefore \frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1} = r$ (say)

$x = x_1 r; \qquad y = y_1 r; \qquad z = z_1 r$

$\therefore$ Any point on OP is (rx_1, ry_1, rz_1)

If this point lies on the cone (1)

Then

$$ar^2x_1^2 + 2hrx_1ry_1 + 2br^2y_1^2 + 2fr^2y_1z_1 + 2gr^2z_1x_1 + 2hr^2x_1y_1 + 2urx_1 + 2vry_1 + 2wrz_1 + d = 0$$
$$r^2(ax_1^2 + 2hx_1y_1 + 2by_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1) + 2r(ux_1 + vy_1 + wz_1) + d = 0$$

Treating the above as an identify.

$$ax_1^2 + 2hx_1y_1 + by_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 = 0 \qquad \text{---(2)}$$

$$ux_1 + vy_1 + wz_1 = 0 \qquad \text{---(3)}$$

$$d = 0 \qquad \text{---(4)}$$

If u,v,w be not all zero, then (3) represents a plane.

$\therefore$ u = v = w = 0.

$\therefore$ (2) becomes

$$ax_1^{\,2} + 2hx_1y_1 + by_1^{\,2} + 2\,fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 = 0$$

$\therefore$ The locus of (x_1, y_1, z_1) is

$$ax^2 + 2hxy + by^2 + 2\,fyz + 2gzx + 2\,fxy = 0$$

Which is a homogeneous equation second degree 2 in x, y, and z.

Bookwork 3: A homogenous equation of second degree in x, y and z always represents a cone with the vertex at the origin.

Solution: Let the homogenous equation of second degree in x, y and z be

$$ax^2 + 2hxy + by^2 + 2\,fyz + 2gzx + 2hxy = 0 \qquad \text{---(1)}$$

Let P (x_1, y_1, z_1) be any point on the surface represented by (1)

$$\therefore\; ax_1^{\,2} + 2hx_1y_1 + by_1^{\,2} + 2\,fy_1z_1 + 2gz_1x_1 + 2hx_1y_1 = 0 \qquad \text{---(2)}$$

Let O be the point (0,0,0)

$\therefore$ The equation of OP is $\dfrac{x}{x_1} = \dfrac{y}{y_1} = \dfrac{z}{z_1} = r$ (say)

$\therefore$ Any point on OP =(x_1r, y_1r, z_1r)

This point lies on (1) , using (2)

$\therefore$ (1) represents a cone whose vertex is at the origin.

13.3. Enveloping cone of a sphere.

Definition:

The locus of the tangent lines to a sphere drawn from a given point is a cone called the enveloping cone of the sphere having the given point as its vertex.

13.4. Examples

Bookwork 4: Find the equation of the enveloping cone of the sphere $x^2 + y^2 + z^2 = a^2$ with its vertex at (x_1, y_1, z_1)

Solution: Let the equation of the sphere be $x^2 + y^2 + z^2 = a^2$ ---(1)

Let the given point be A (x_1, y_1, z_1)

Let P (x,y,z) be any point on a tangent drawn from A to the sphere

The coordinates of a point dividing AP in the ratio m=1 is

$$\left(\frac{mx + x_1}{m+1}, \frac{my + y_1}{m+1}, \frac{mz + z_1}{m+1}\right)$$

If this lies on (1)

Then $\dfrac{(mx + x_1)^2}{(m+1)^2} + \dfrac{(my + y_1)^2}{(m+1)^2} + \dfrac{(mz + z_1)^2}{(m+1)^2} = a^2$

$$(mx + x_1)^2 + (my + y_1)^2 + (mz + z_1)^2 = a^2(m+1)^2$$

$$m^2(x^2 + y^2 + z^2 - a^2) + 2m(xx_1 + yy_1 + zz_1 - a^2) + x_1{}^2 + y_1{}^2 + z_1{}^2 - a^2 = 0 \quad \text{---(2)}$$

The line AP is a tangent line to (1)

$\therefore$ The roots are equal.

$$\therefore\ 4(xx_1 + yy_1 + zz_1 - a^2) = 4(x^2 + y^2 + z^2 - a^2) \times (x_1{}^2 + y_1{}^2 + z_1{}^2 - a^2) = 0$$

$$\therefore\ (xx_1 + yy_1 + zz_1 - a^2) = (x^2 + y^2 + z^2 - a^2) \times (x_1{}^2 + y_1{}^2 + z_1{}^2 - a^2) = 0$$

Which is the required equation.

Problems (1) Find equation of the line whose vertex is at (α, β, γ) and base

$y^2 = 4ax,\ Z = 0$

Solution: Any line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{(---1)}$$

Equation of the base is $y^2 = 4ax,\ Z = 0$ (---2)

The line (1) meets the plane Z =0

$\therefore$ Put Z =0 in (1)

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$$

$$x - \alpha = \frac{-l}{n}\gamma; \quad y - \beta = \frac{-m}{n}\gamma$$

$$x = \alpha \frac{-l}{n}\gamma ; \qquad y = \beta \frac{-m}{n}\gamma$$

Using in (1)

$$\left(\beta - \frac{m}{n} r\right)^2 + 4a\left(\alpha - \frac{l}{n} r\right) \qquad \text{(---3)}$$

Eliminate l,m,n between (1) and (3)

$$\left(\beta - \frac{y-\beta}{z-\gamma} r\right)^2 + 4a\left(\alpha - \frac{x-\alpha}{z-\gamma} r\right)$$

$$\frac{(\beta z - y\gamma)^2}{(z-r)^2} + 4a\frac{(\alpha z - x\gamma)}{(z-r)}$$

Multiply both sides by $(z-r)^2$

$$\therefore \qquad (\beta z - y\gamma)^2 = +4a(\alpha z - x\gamma)(z-r)$$

2. Find the enveloping cone of the sphere $x^2 + y^2 + z^2 + 2x - 2y - 2 = 0$ with its vertex (1,1,1)

Solution:

$$S = x^2 + y^2 + z^2 + 2x - 2y - 2$$

$$T = xx_1 + yy_1 + zz_1 + x + x_1 - y - y_1 - z - z_1 - 2$$

Here $(x_1, y_1, z_1) = (1,1,1)$

$$= x + y + z + x + 1 - y - 1 - z - 1 - 2$$

$$= 2x + z - 2$$

$$S_1 = 1^2 + 1^2 + 1^2 + 2 - 2 - 2$$

$$= 1$$

FORMULA $T^2 = SS_1$

$$(2x + z - 2)^2 = x^2 + y^2 + z^2 + 2x - 2y - 2$$

$$\therefore 4x^2 + z^2 + 4 + 4xz - 4z - 8x - x^2 - y^2 - z^2 - 2x + 2y + 2 = 0$$

$$\therefore 3x^2 - y^2 + 4xz - 10x + 2y - 4z + 6 = 0$$

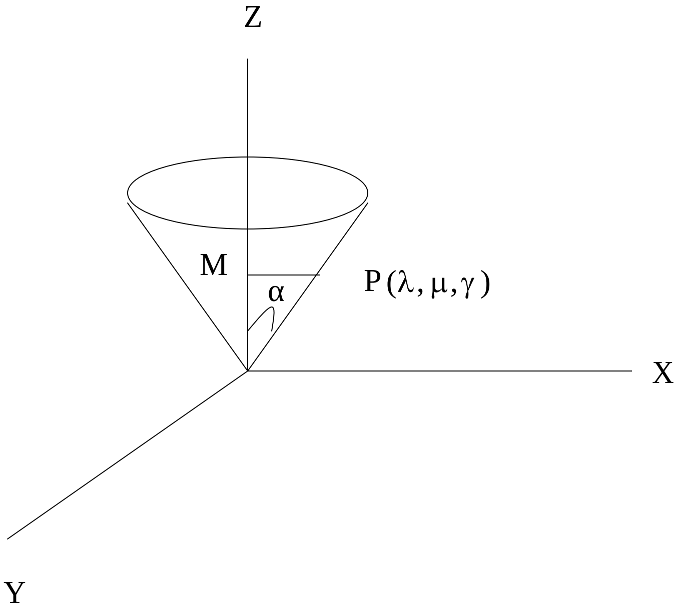
13.5. Right circular cone

Definition:

A right circular cone is a surface generated by a straight line which passes through a fixed point and makes a constant angle with a fixed line. The fixed point is called the vertex of the cone. The constant angle is called the semi-vertical angle of the cone. The fixed line is called the axis of the cone.

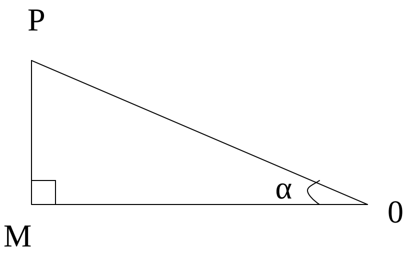
13.6. Examples

1) Standard Equation of a right circular cylinder.



Let P (λ, μ, γ) be any point on the cone

Given that $P\hat{O}Z = \alpha$.



Draw PM perpendicular to OZ in ΔOPM

$$\tan\alpha = \frac{PM}{OM}$$

$\therefore PM^2 = OM^2 \tan^2\alpha$ ---(1)

But $PM^2 = OP^2 - OM^2$

$$= \lambda^2 + \mu^2 + \gamma^2 - \theta \text{ projection of OP on OZ}$$

$$= \lambda^2 + \mu^2 + \gamma^2 - \gamma^2$$

$$PM^2 = \lambda^2 + \mu^2$$

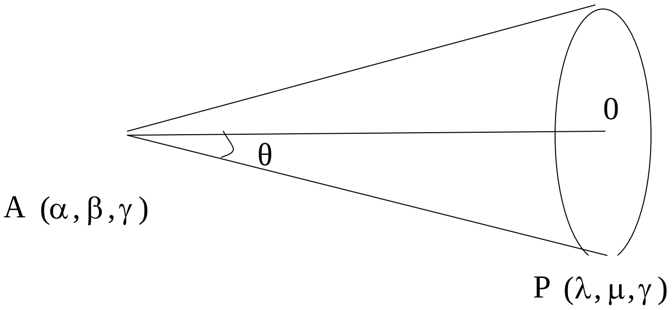
Using in (1)

$$\lambda^2 + \mu^2 = \gamma^2 \tan^2 \alpha$$

$\therefore$ The locus (λ, μ, γ) is $x^2 + y^2 = z^2 \tan^2 \alpha$

(2) Find the equation of a right circular cone with its vertex at (α, β, γ), it's axis the line $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and its semivertical angle θ, l, m, n being d.r's of the axis.

Solution:



Let A (λ, μ, γ) be the vertex of the curve

Let P (λ, μ, γ) be any point on the cone

Let AP wake an angle θ with the axis AO of the cone.

The d.r's of AP are $\lambda - \alpha, \mu - \beta, \gamma - \gamma$

The d.r's of OA are *l*,m,n

$$\cos\theta = \frac{l(\lambda - \alpha) + m(\mu - \beta) + n(\gamma - \gamma)}{\sqrt{l^2 + m^2 + n^2}\sqrt{(\lambda - \alpha)^2 + (\mu - \beta)^2 + (\gamma - \gamma)^2}}$$

$$\therefore l(\lambda-\alpha)+m(\mu-\beta)+n(\gamma-\gamma)=\cos\theta\left[\sqrt{l^2+m^2+n^2}\ \sqrt{(\lambda-\alpha)^2+(\mu-\beta)^2+(\gamma-\gamma)^2}\right]$$

Squaring both sides

$$[l(\lambda-\alpha)+m(\mu-\beta)+n(\gamma-\gamma)]^2=\cos^2\theta\left[\left(l^2+m^2+n^2\right)\left((\lambda-\alpha)^2+(\mu-\beta)^2+(\gamma-\gamma)^2\right)\right]$$

$\therefore$ The focus (λ,μ,γ) is

$$[l(\lambda-\alpha)+m(\mu-\beta)+n(\gamma-\gamma)]^2=\cos^2\theta\left[\left(l^2+m^2+n^2\right)\left((\lambda-\alpha)^2+(\mu-\beta)^2+(\gamma-\gamma)^2\right)\right]$$

Which is the required equation of the cylinder

(3) Find the equation of the right circular cone whose vertex is the point (1,1,1) the axis is the line $\frac{x-1}{-1}=\frac{y-1}{2}=\frac{z-1}{3}$ and semi vertical angle is 30^0

Solution: Let P (λ,μ,γ) be any point on the cone the vertex of the cone is A(1,1,1)

$\therefore$ The d.r's of AP are $\lambda-1,\mu-1,\gamma-1$

The d.r's of the axis are -1,2,3

$$\cos\theta=\frac{-1(\lambda-1)+2(\mu-1)+3(\gamma-1)}{\sqrt{1+4+9}\ \sqrt{(\lambda-1)^2+(\mu-1)^2+(\gamma-1)^2}}$$

$$\frac{\sqrt{3}}{2}=\frac{-\lambda+1+2\mu-2)+3\gamma-3)}{\sqrt{14}\ \sqrt{(\lambda-1)^2+(\mu-1)^2+(\gamma-1)^2}}$$

$$=\frac{-\lambda+2\mu+3\gamma-4}{\sqrt{14}\ \sqrt{(\lambda-1)^2+(\mu-1)^2+(\gamma-1)^2}}$$

$$\therefore\sqrt{3}\sqrt{14}\ \sqrt{(\lambda-1)^2+(\mu-1)^2+(\gamma-1)^2}=(-\lambda+2\mu+3\gamma-4)$$

Squaring both sides

$$42\left[(\lambda-1)^2+(\mu-1)^2+(\gamma-1)^2\right]=(-\lambda+2\mu+3\gamma-4)^2$$

The focus of (λ,μ,γ) is

$$42\left[(x-1)^2+(y-1)^2+(z-1)^2\right]=(-x+2y+3z-4)^2$$

On simplification, the equation of the right circular cone is

$$19x^2+13y^2+3z^2+8xy+12xz-24yz-58x-10y+6z+31=0$$

(4) Find the condition that the general equation of the second degree may represent a cone

Solution:

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0 \quad \text{---(1)}$$

Represent a cone with vertex at the point (x_1,y_1,z_1)

Shifting the origin to the point (x_1,y_1,z_1) (1) becomes

$$\begin{aligned} a(x+x_1)^2+b(y+y_1)^2+c(z+z_1)^2+2f(y+y_1)(z+z_1)+2g(z+z_1)(x+x_1) \\ +2h(x+x_1)(y+y_1)+2a(x+x_1)+2v(y+y_1)+2w(z+z_1)+d=0 \end{aligned}$$

$$\begin{aligned} &ax^2+by^2+cz^2+2fyz+2gzx+2hxy+ \\ &\quad 2\left[x(ax_1+hy_1+gz_1+u)+y(hx_1+by_1+fz_1+v)+z(gx_1+fy_1+cz_1+w)\right] \\ &\quad\quad +ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1+2wz_1+d=0 \end{aligned}$$

This equation is referred to (x_1,y_1,z_1) which is the new origin

$\therefore$ $ax_1+hy_1+gz_1+u=0$ ---(2)

$hx_1+by_1+fz_1+v=0$ ---(3)

$gx_1+fy_1+cz_1+w=0$ ---(4)

$$ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1+2wz_1+d=0 \quad \text{---(5)}$$

This can be written as

$$x_1(ax_1+hy_1+gz_1+u)+y_1(hx_1+by_1+fz_1+v)+z_1(gx_1+fy_1+cz_1+w)+ux_1+vy_1+wz_1+d=0$$

$\therefore$ using (2), (3), (4); $ux_1+vy_1+wz_1+d=0$ ---(6)

Eliminate x_1, y_1, z_1 between (2), (3), (4),(6)

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

Which is the required condition.

(5) Prove that the equation

$7x^2 + 2y^2 + 2z^2 + 10zx + 10xy + 26x - 2y + 2z - 17 = 0$ represents a cone/

Proof: The given equation is of the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

$a = 7;$ $b = 2;$ $c = 2;$ $2f = 0$ $2g = -10$ $2h = 10$

$f = 0$ $g = -5$ $h = 5$

$2u = 26$ $2v = -2$ $2w = 2$ $d = -17$

$u = 13$ $v = -1$ $w = -1$

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = \begin{vmatrix} 7 & 5 & -5 & 13 \\ 5 & 2 & 0 & -1 \\ -5 & 0 & 2 & 1 \\ 13 & -1 & 1 & -17 \end{vmatrix}$$

$$= 7\begin{vmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & -17 \end{vmatrix} - 5\begin{vmatrix} 5 & 0 & -1 \\ -5 & 2 & 1 \\ 13 & 1 & -17 \end{vmatrix} - 5\begin{vmatrix} 5 & 2 & -1 \\ -5 & 0 & 1 \\ 13 & -1 & -17 \end{vmatrix} + 13\begin{vmatrix} 5 & 2 & 0 \\ -5 & 0 & 2 \\ 13 & -1 & 1 \end{vmatrix}$$

$= 0$ (on simplification)

$\therefore$ The given expression represents cone.

Method 2: To find the vertex of a cone where f(x_1,y_1,z_1) as a homogenous function of degree in variable x,y,z,t find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial t}$ and square to zero after substituting t = 1.

Solve fx = 0, fy = 0, fz = 0, ft = 0 and t = 1, if the equations are consistent, then the given equation represents a cone and the value of (x,y,z) denote the vertex of the cone.

(1) Prove that the equation

$2x^2 + 2y^2 + 7z^2 - 10yz + 10zx + 2x + 2y + 2y + 26z - 17 = 0$ represents a cone with vertex at (2,2,1)

Solution:

Let f(x,y,z) = $2x^2 + 2y^2 + 7z^2 - 10yz + 10zx + 2x + 2y + 2y + 26z - 17 = 0$

f (x,y,z,t) = $2x^2 + 2y^2 + 2z^2 - 10yz - 10zx + 2xt + 2yt + 26zt - 17t^2$

$$\frac{\partial f}{\partial x} = 4x - 10z + 2t; \qquad \frac{\partial f}{\partial y} = 4y - 10z + 2t;$$

$$\frac{\partial f}{\partial z} = 14z - 10y + 26t - 10x$$

$$\frac{\partial f}{\partial t} = 2x + 2y + 26z - 34t$$

Put t = 1, in the above partial derivation and equate these to Zero.

$$\therefore 4x - 10z + 2 = 0 \quad ie \quad 2x - 5z + 1 = 0 \qquad \text{---(1)}$$

$$4y - 10z + 2 = 0 \quad \therefore \quad 2y - 5z + 1 = 0 \qquad \text{---(2)}$$

$$-10x - 10y + 14z + 26 = 0 \quad ie \quad -5x - 5y + 7z + 13 = 0 \qquad \text{---(3)}$$

$$2x + 2y + 26z - 34t = 0$$

$$\therefore\ x + y + 13z - 17 = 0 \qquad \text{---(4)}$$

Solve (1), (2) , (3)

2x – 5z + 1 = 0

2y – 5z + 1 = 0

Sub $2x - 2y = 0$

$\therefore\ x = y$ ---(5)

Put x = y in (3) $-5y - 5y + 7z + 13 = 0$

$\therefore$ $-10y + 7z + 13 = 0$ ---(6)

But (2) is $2y - 5z + 1 = 0$ ---(2)

Solve (2) and (6); (2) x 5; $10y - 25z + 5 = 0$

(6) is $-10y + 7z + 13 = 0$

Add $-18z + 18 = 0$

$\therefore\ z = 1$

Put $z = 1$ in (2) $2y - 5 + 1 = 0$

$2y - 4 = 0$

$2y = 4$

$y = 2$

$\therefore\ x = 2$

Hence x = 2 , y = 2, z = 1

Using in (4)

LHS = $x + y + 13z - 17$

= $2 + 2 + 13 - 17$

= 0 = RHS

$\therefore$ Equations (1), (2), (3), (4) are consistent

$\therefore$ The given equation represents a sphere with the vertex (2,2,1)

13.7. Cylinder

The Cylinder is the surface generated by a variable straight line which remains parallel to a fixed straight line and satisfies one more condition, ie. It may intersect a given curve or it may touch a given surface.

13.8. Examples

(1) To find the equation of a cylinder whose generators are parallel to the line

$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ and base the conic $ax^2 + by^2 + 2fyz + 2hxy + 2gx + 2fy + c = 0,\ z = 0$

Solution: The equation of the base is $ax^2 + by^2 + 2fyz + 2hxy + 2gx + 2fy + c = 0,\ z = 0$

--(1)

Let (x_1,y_1,z_1) be any point on the generator, which is parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

$\therefore$ Equation of the generator is $\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ ---(2)

This meets the plane z = 0

Put $z = 1$ in (2)

$$\therefore \frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z_1}{n}$$

$$\therefore \frac{x - x_1}{l} = \frac{z_1}{n}; \qquad y - y_1 = \frac{-m}{n} z_1$$

$$x = \frac{-l}{n} z_1 + x_1; \qquad y = y_1 - \frac{m}{n} z_1$$

Using in (1)

$$a\left(x_1 - \frac{l}{n}z_1\right)^2 + b\left(y_1 - \frac{m}{n}z_1\right)^2 2h\left(x_1 - \frac{l}{n}z_1\right)\left(y_1 - \frac{m}{n}z_1\right) + 2g\left(x_1 - \frac{l}{n}z_1\right) + 2f\left(y_1 - \frac{m}{n}z_1\right) + c = 0$$

$$a(nx_1 - lz_1)^2 + b(ny_1 - mz_1)^2 + 2h(nx_1 - lz_1)(ny_1 - mz_1) + 2gn(nx_1 - lz_1) + 2fn(ny_1 - mz_1) + cn^2 = 0$$

Which is the equation of the cylinder.

(1) Find the equation of a cylinder whose generators are parallel to the line $x = -\frac{y}{2} = \frac{z}{3}$

and whose guiding curve is the ellipse $x^2 + 2y^2 = 1, z = 3$

Solution: The equation of the guiding curve is $x^2 + 2y^2 = 1, z = 3$ ---(1)

Let (x_1,y_1,z_1) be any point on the generator. Since the generator is parallel to the

line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$,

Equation of the generator is $\frac{x - x_1}{1} = \frac{y - y_1}{-2} = \frac{z - z_1}{3}$

This meets the plane z =3

$$\therefore \qquad \frac{x - x_1}{1} = \frac{y - y_1}{-2} = \frac{3 - z_1}{3}$$

$$\therefore \qquad x = x_1 + \frac{3 - z_1}{3}, \qquad y = y_1 - \frac{2}{3}(3 - z_1)$$

$$\frac{3x_1 + 3 - z_1}{3}; \qquad y = \frac{3y_1 - 6 + 2z_1}{3}$$

Using in (1) $x^2 + 2y^2 = 1$, we get

$$\left(\frac{3x_1 + 3 - z_1}{3}\right)^2 + 2\left(\frac{3y_1 - 6 + 2z_1}{3}\right)^2 = 1$$

$$(3x_1 + 3 - z_1)^2 + 2(3y_1 - 6 + 2z_1)^2 = 9$$

On simplification

The focus of (x_1,y_1,z_1) is $3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0$

13.9 Right circular cylinder

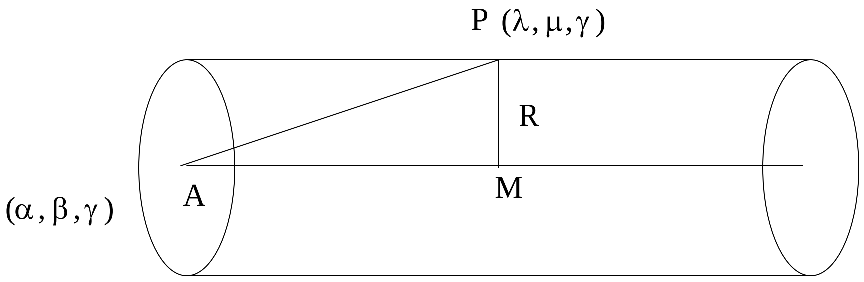
The right circular cylinder is the surface generated by a straight line which is parallel to a fixed line and is at a constant distance from it.

The variable line is called the generator of the cylinder. The fixed line is called the axis of the cylinder. The constant distance is called the radius of the cylinder.

13.10. Examples

(1) Find the equation of a right circular cylinder whose axis is the line

$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ and where radius is R units.



The equation of the axis are $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ ----(1)

Let P (λ, μ, γ) be any point on the cylinder

Draw PM perpendicular to the axis.

PM = R

Join PA where A = (α, β, γ)

AM = The projection of AP on the axis of the cylinder

$$= \frac{l(\lambda-\alpha)+m(\mu-\beta)+n(\gamma-\gamma)}{\sqrt{l^2+m^2+n^2}}$$

From ΔAPM, $AP^2 + AM^2 + PM^2$

$$(\lambda-\alpha)^2+(\mu-\beta)^2+(\gamma-\gamma)^2 = \frac{[l(\lambda-\alpha)+m(\mu-\beta)+n(\gamma-\gamma)]^2}{l^2+m^2+n^2}$$

$$\therefore \left(l^2+m^2+n^2\right)\left[(\lambda-\alpha)^2+(\mu-\beta)^2+(\gamma-\gamma)^2\right] = [l(\lambda-\alpha)+m(\mu-\beta)+n(\gamma-\gamma)]^2 + R^2$$

$$= [l(\lambda-\alpha)+m(\mu-\beta)+n(\gamma-\gamma)]^2 + R^2\left(l^2+m^2+n^2\right)$$

$\therefore$ The focus of (λ, μ, γ) is $\left[(\lambda-\alpha)^2+(\mu-\beta)^2+(\gamma-\gamma)^2\right]\left(l^2+m^2+n^2\right) =$

$$= [l(\lambda-\alpha)+m(\mu-\beta)+n(\gamma-\gamma)]^2 + R^2\left(l^2+m^2+n^2\right)$$

which is the required equation.

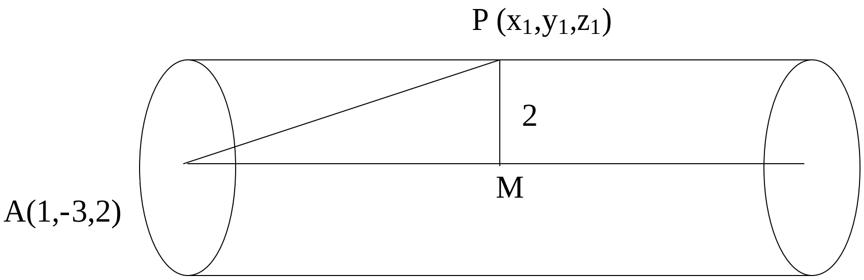
(1) The radius of a normal section of a right circular cylinder is 2 units, the axis lie along the line $\frac{x-1}{1}=\frac{y+3}{-1}=\frac{z-2}{5}$. Find its equation

Solution: The equation of the axis is

$$\frac{x-1}{1}=\frac{y+3}{-1}=\frac{z-2}{5}$$

The d.r's are 1, -1, 5

$\therefore$ Its d.r's are $\frac{1}{\sqrt{27}},\frac{-1}{\sqrt{27}},\frac{5}{\sqrt{27}}$



Let P(x_1,y_1,z_1) be any point on the cylinder

$$AP=\sqrt{(x_1-2)^2+(y_1+3)^2+(z_1-2)^2}$$

R= 2 = Radius of the cylinder

AM = The projection of AP on the axis

$$=\frac{1}{\sqrt{27}}\left[(x_1-1)+(y_1+3)+(z_1-2)\right]$$

$$=\frac{1}{\sqrt{27}}\left[x_1+y_1+z_1\right]$$

$$AM^2=\frac{1}{27}\left[x_1+y_1+z_1\right]^2$$

In $\Delta APM,\quad AP^2+AM^2+PM^2$

$$(x_1-1)^2+(y_1+3)^2+(z_1-2)^2=\frac{1}{27}(x_1+y_1+z_1)^2+2^2$$

$$27\left[(x_1-1)^2+(y_1+3)^2+(z_1-2)\right]^2=(x_1+y_1+z_1)^2+108$$

Simplifying, the focus of (x_1,y_1,z_1) is

$$29x^2+29y^2+29z^2+2xy-2xz+2yz-30x+150y-93z+75=0$$

2. Find the equation of the right circular cylinder whose guiding circle is

$$x^2+y^2+z^2=29; \quad x-y+z=3$$

Solution:

The circle is nothing but the plane section of the sphere $x^2+y^2+z^2=9$, by the plane

$x-y+z=3$

NP = Radius of the circle.

$$=\sqrt{(radius\ of\ the\ sphere)^2-(perpendicular\ dis\tan ce\ from\ the\ centre\ of\ the\ sphere\ to\ the\ plane)^2}$$

$$=\sqrt{r^2-p^2} \qquad \text{---(1)}$$

Radius of the sphere = 3

Centre of the sphere = (0,0,0)

p = perpendicular distance from the centre (0,0,0) to the plane x – y + z -3 =0

$$=\pm\frac{(-3)}{\sqrt{1^2+1^2+1^2}}=\frac{3}{\sqrt{3}}=\frac{\sqrt{3}\sqrt{3}}{\sqrt{3}}=\sqrt{3}$$

$\therefore$ using in (1)

$$R=\sqrt{3^2-(\sqrt{3})^2}=\sqrt{6}$$

M is the centre of the circle.

The d.r's of the normal to the plane are (1,-1,1).

Equation of PMB $\frac{x-0}{1}=\frac{y+0}{-1}=\frac{z-0}{1}$

$$\frac{x}{1}=\frac{y}{-1}=\frac{z}{1}=\lambda \text{ (say)}$$

$$x = \lambda,\ \ y = -\lambda,\ z = \lambda$$

Any point on PM = $\left(\lambda,\ -\lambda,\ \lambda\right) = M$ (say)

This lies on the plane x – y + z = 3.

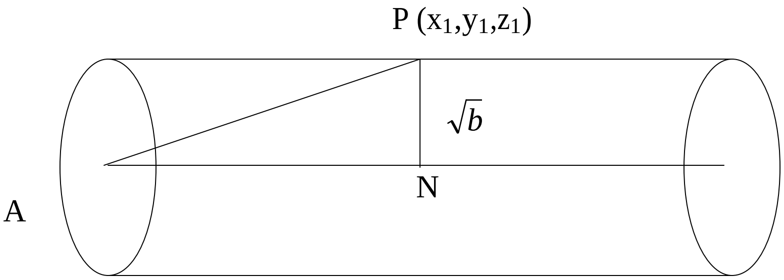
$$\therefore \qquad \lambda + \lambda + \lambda = 3$$

$$3\lambda = 3$$

$$\lambda = 1$$

$\therefore$ Centre of the circle is (1,1,1)

Step 2: Let P (x,y,z) be any point on the cylinder



A(0,0,0) P (x_1,y_1,z_1)

$$AP^2 = x_1^{\ 2} + y_1^{\ 2} + z_1^{\ 2}$$

$$NP = \sqrt{6} \qquad \therefore NP^2 = 6$$

AN = The projection of AP on AN

$$= \frac{1}{\sqrt{3}}(x_1 + y_1 + z_1)$$

In ΔAPN,

$$AP^2 = AN^2 + NP^2$$

$$x_1^{\ 2} + y_1^{\ 2} + z_1^{\ 2} = 6 + \left[\frac{1}{\sqrt{3}}(x_1 + y_1 + z_1)\right]^2$$

$$=6+\frac{1}{\sqrt{3}}(x_1 + y_1 + z_1)^2$$

$$3x_1^{\,2} + 3y_1^{\,2} + 3z_1^{\,2} = 18 + x_1^{\,2} + y_1^{\,2} + z_1^{\,2} + 2x_1y_1 + 2y_1z_1 + 2z_1x_1$$

$$2x_1^{\,2} + 2y_1^{\,2} + 2z_1^{\,2} - 2x_1y_1 - 2y_1z_1 - 2z_1x_1 - 18 = 0$$

$\div 2;$

$$x_1^{\,2} + y_1^{\,2} + z_1^{\,2} - x_1y_1 - y_1z_1 - z_1x_1 - 9 = 0$$

The focus of (x_1,y_1,z_1) is

$$x^2 + y^2 + z^2 - xy - yz - zx - 9 = 0$$

13.11. Let us sum up.

We have leant the different types of cone, cylinder and the dimension their equations in different – forms.

13.12. Check your progress

(1) Find the equation of a right circular cylinder of radius 2 and which axis is x

(2) What is the general equating a give passing through the axis

13.13. Lesson End Activities

Cone:

1. Find the equation to the cone whose vertex is the origin and which passes through the curve of intersection of

   (i) $ax^2 + by^2 + cz^2 = 1,\ lx + my + nz = p$

   Ans: $p(ax^2 + by^2 + cz^2) = (\ lx + my + nz)^2$

   (ii) $ax^2 + by^2 = 2z;\ \ lx + my + nz = p$

   Ans: $p(ax^2 + by^2) = 2z(\ lx + my + nz)$

2. If $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is a generator of the cone, represented by f(x,y,z) = 0, then prove that f(*l*,m,n) = 0

3. Find the equation of the cone whose vertex is the point (α, β, γ) and whose generating line pass through the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$

   Ans: $\frac{1}{a^2}(\alpha z - xr)^2 + \frac{1}{b^2}(\beta z - yv)^2 = (z - v)^2$

4. Prove that the equation of the cone whose vertex is the point (1,1,0) and whose generating curve $y = 0,\ x^2 + z^2 + = 4\, is\ \ x^2 + 3y^2 + z^2 - 2xy + 8y - 4 = 0$

5. Find the equation of a right circular cone whose vertex is at the origin, whose axis is the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ and which has a semi-vertical angle of 60^0

   Ans : $19x^2+13y^2+3z^2-24yz-12zx-8xy=0$

6. Prove that the equation $7x^2+2y^2+2z^2-10zx+10xy+26x-2y+2z-17=0$ represents a cone whose vertex is at (1, -2, 2)

7. Find the equation of a right circular cone whose axis is the line x = y = z and the generator is 2x = y = -3z (APRIL 2004; Bharathiyar)

   (Hint: To find semi vertical angle: The d.r's of the axis are 1 , 1, 1

   a_1, b_1, c_1

   The d.r's of the generator are $\frac{1}{2}, 1, \frac{-1}{3}$

   a_2, b_2, c_2

   The semi-vertical angle is given by $\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

8. Find the general equation of the cone which touches the coordinate planes (NOV 2000, Bharathiyar)

13.14. Points for discussion

1. Find the equation of the right circular cone with vertex at the origin semi vertical angle 30^0 and the line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ being the axis of the conic (April 2004, Bharathiar)
2. Find the equation of the cone with the vertex at (1,1,1) and base curve $x^2+2y^2=1, z=0$ (April 2004; Bharathiar)
3. The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinates axes in A,B,C. Prove that the equation to the cone generated by lines drawn from 0 to meet the circle ABC is $\sum yz\left(\frac{b}{c}+\frac{c}{b}\right)=0$.

4. Find the equation of the cylinder whose generators are parallel to the line x = y = z and whose guiding curve is the circle.

   $x^2 + y^2 + z^2 - 2x - 3 = 0, 2x + y + 2z = 0$

   Ans: $17x^2 + 18y^2 + 17z^2 - 18zy - 16zx - 18yz - 40x + 10y + 20z - 75 = 0$

5. Find the equation of the right circular cylinder of radius 2 whose axis passes through (1,2,3) and has direction cosines proportional to 2, -3, 6

   Ans: $45x^2 + 40y^2 + 13z^2 + 12xy + 36yz - 24zx - 42x - 280y - 126z + 294 = 0$

6. Find the equation of the cylinder whose generators are parallel to the line

   $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is $x^2 + 2y^2 = 1, z = 3$

   Ans: $3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0$

7. Find the equation of the right circular cylinder of radius 3 whose axis passes through (2,3,4) and has direction cosines proportional to (2,1,-2)

   Ans: $5x^2 + 8y^2 + 5z^2 - 4xy + 4yz + 8xz - 40x - 56y - 68z + 179 = 0$

8. Find the equation of a right circular cylinder of radius 3 with axis

   $\frac{x+2}{3} = \frac{y-4}{6} = \frac{z-1}{2}$ (April 2005, Bharathiar)

13.15 References

Analytical Geometry of Three Dimension by N.P. Bali

Lesson – 14

Contents

14.0 Aims and Objectives

In this lesson we are going to study about use definition coin cord which is a new concept, the standard equation of coin cord, tangent plane, enveloping cone, director sphere.

14.1 Concicoids

The general equation of second degree in x,y,z ie.

$$f(x,y,z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

represents a locus called a coincoid or a quadric.

Note: (1) The equation of a conicoid contains only line disposable constants

(2) We can reduce this equation to some standard form by the suitable change of the axes.

In other words, we know a conic is intersected by a straight line at two points.

$\therefore$ The curve of intersection of a plane and a quadric surface is a conic.

$\therefore$ Quadric surfaces are called coincoids

Centre of the coincoid: If every chord of the conicoid passes through the origin , it is called the centre of the coincoid central conicoid. The conic having a centre is called a central conicoid.

14.2. Nature of a conicoid:

The standard equation of a central conicoid is $ax^2 + by^2 + cz^2 = 1$

This represents i) an ellipsoid if a,b,c are all positive ii) a hyperboloid of one sheet if two are positive and the one is negative. (iii)a hyperboloid two sheets if two are negative and one positive. (iv) a virtual ellipsoid if all are negative.

Note (1). The conicoid $ax^2 + by^2 + cz^2 = 1$ has origin as centre

(2). The equation $ax^2 + by^2 + cz^2 = 1$ is called the standard equation of the conicoid.

14.3. Enveloping cone

The locus of the tangent lines drawn from a given point to a given conicoid is a cone called the enveloping cone of the conicoid or tangent cone of the concoid having the given point as its vertex.

14.4. Examples

(1) Find the equation of the enveloping cone of the conic coid $ax^2 + by^2 + cz^2 = 1$ with its vertex at A(x_1,y_1,z_1)

Solution: The equation of the conicoid is

$$ax^2 + by^2 + cz^2 = 1 \qquad \text{---(1)}$$

Let A be the point (x_1,y_1,z_1)

Equation of any line through A(x_1,y_1,z_1) is

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = R \text{ (say)} \qquad \text{---(2)}$$

Any point on (2) is $(x_1 + Rl), y_1 + Rm, z_1 + Rn)$

If this point lies on (1), then

$$a(x_1 + Rl)^2 + b(y_1 + Rm)^2 + c(z_1 + Rn)^2 = 1$$

$$R^2(al^2 + bm^2 + cn^2) + 2R(alx_1 + bmy_1 + cnz_1) + (ax_1{}^2 + by_1{}^2 + cz_1{}^2) = 0$$

The line (2) is a tangent line to (1)

$\therefore$ The above quadratic equation has equal roots.

$$\therefore\ 4(alx_1+bmy_1+cnz_1)^2 = 4(al^2+bm^2+cn^2)\lambda(ax_1^2+by_1^2+cz_1^2-1) \qquad \text{---(3)}$$

Eliminate l,m,n between (2) and (3)

$$[a(x-x_1)x_1+b(y-y_1)y_1+c(z-z_1)z_1]^2 = [a(x-x_1)^2+b(y-y_1)^2+c(z-z_1)^2](ax_1^2+by_1^2+cz_1^2-1)$$

$$[(axx_1+byy_1+czz_1-1)-(ax_1^2+by_1^2+cz_1^2-1)]^2 =$$

$$[(ax^2+by^2+cz^2)-2(axx_1+byy_1+czz_1-1)+(ax_1^2+by_1^2+cz_1^2-1)] \quad \text{--(4)}$$

$$\times(ax_1^2+by_1^2+cz_1^2-1)$$

Let T = $axx_1+byy_1+czz_1-1$

S = $ax^2+by^2+cz^2$ -1

S_1= $ax_1^2+by_1^2+cz_1^2-1$

Using in (4)

$(T-S_1)^2=[S-2T+S_1]S_1$

$T^2-2TS_1+S_1^2=SS_1-2TS_1+S_1^2$

$T^2=SS_1$

$$[axx_1+byy_1+czz_1-1]^2 = (ax^2+by^2+cz^2-1)(ax_1^2+by_1^2+cz_1^2-1)$$

Which is the required equation.

Tangent plane

(2) Find the equation of the tangent plane at (x_1,y_1,z_1) of the conicoid $ax^2+by^2+cz^2=1$

Solution:

The equation of the central conicoid is $ax^2+by^2+cz^2=1$ ---(1)

Equations of any line through (x_1,y_1,z_1) is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=R\,(\text{say})$$

Any point on the line is

$$x=x_1+Rl,\ y=y_1+Rm;\ z=z_1+Rn$$

i.e $\quad (x_1+Rl,\ y_1+Rm;\ z_1+Rn)$

If this lies on (1) then

$$a(x_1+Rl)^2+b(y_1+Rm)^2+c\ (z_1+Rn)^2=1$$

$$a[x_1^2+2Rl\,n_1+R^2l^2]+b[y_1^2+2\,y_1mR+m^2R^2]+c[z_1^2+2\,y_1nR+n^2R^2]=1$$

$$R^2(al^2+bm^2+cn^2)+2R(lx_1+my_1+nz_1)+ax_1^{\ 2}+by_1^{\ 2}+cz_1^{\ 2}-1=0$$

But (x_1,y_1,z_1) lies in (1)

$$\therefore \qquad ax_1^{\ 2}+by_1^{\ 2}+cz_1^{\ 2}-1=0$$

$$\therefore R^2(al^2+bm^2+cn^2)+2R(lx_1+my_1+nz_1)=0$$

This is a quadratic equation in R.

Whose roots are R_1 and R_2

Since the line (2) is a tangent line in quadratic equation

$$ax^2+by+c=0,$$

$$b^2-4ac=0$$

But c = 0

$$\therefore b^2=0$$
$$b=0$$

The roots are equal

$\therefore$ coefficient of R = 0

$$\therefore lx_1+my_1+nz_1=0 \qquad \text{---(3)}$$

Eliminate *l*,m,n between (2) and (3)

$$(x-x_1)x_1+(y-y_1)y_1+(z-z_1)z_1=0$$

$$xx_1+yy_1+zz_1=ax_1^{\ 2}+by_1^{\ 2}+cz_1^{\ 2}$$

$$xx_1+yy_1+zz_1=1$$

Which is the required equation of the tangent plane at (x_1,y_1,z_1)

Condition for tangency

(3) Find the condition for the plane *l*x+my+nz = P to be a tangent plane to the conicoid

$$ax_1^{\ 2}+by_1^{\ 2}+cz_1^{\ 2}=1$$

Solution: Let the line *l*x+my+nz = P ---(1) touch the conicoid of (x_1,y_1,z_1)

$\therefore$ Equation of the tangent plane at (x_1,y_1,z_1) is $axx_1+byy_1+czz_1=1$---(2)

But equations (1) and (2) represent the same tangent plane

$\therefore$ their corresponding coefficients are proportional

$$\therefore \frac{ax_1}{l} = \frac{by_1}{m} = \frac{cz_1}{n} = \frac{1}{p}$$

$$(i) \quad (ii) \quad (iii) \quad (iv)$$

From (i) and (iv)

$$\therefore \frac{ax_1}{l} = \frac{1}{p}, \quad y_1 = \frac{m}{bp}; \qquad z_1 = \frac{n}{p}; \qquad x_1 = \frac{1}{ap} \quad ;$$

But (x_1,y_1,z_1) lies on $ax_1^2 + by_1^2 + cz_1^2 = 1$

$$\therefore \quad a.\frac{l^2}{a^2p^2} + b.\frac{m^2}{b^2p^2} + c.\frac{n^2}{c^2p^2} = 1$$

$$\frac{l^2}{ap^2} + \frac{m^2}{bp^2} + \frac{n^2}{cp^2} = 1$$

$$\frac{1}{p^2}\left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right) = 1$$

$$\therefore \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2$$

Which is the required condition.

Cor 1: The point of contact is $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$

Cor 2: Find the equation of two tangent planes to the central conicoid $ax^2 + by^2 + cz^2 = 1$ which are parallel to the plane lx+my+nz = 0

Solution: Equation of the plane parallel to the plane lx+my+nz = 0 is

$$lx+my+nz = P \qquad \text{--- (1)}$$

(1) touches the conicoid $ax^2 + by^2 + cz^2 = 1$

$$\therefore p^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}$$

$$\therefore p = \pm\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$$

Using in (1), Equations two tangent planes are

$$lx+my+nz = \pm\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$$

14.5. Director Sphere

The locus of the point of intersection of three mutually perpendicular tangent planes to a conicoid is called a director sphere.

Find the equation of the director sphere of the coincoid $ax^2+by^2+cz^2=1$ (April 2006, Bharathiyar)

Proof: The equation of the central conicoid is $ax^2+by^2+cz^2=1$

Let the three mutually perpendicular planes be

$$1_l x+m_1 y+n_1 z = \pm\sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}} \qquad \text{---(2)}$$

$$1_2 x+m_2 y+n_2 z = \pm\sqrt{\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}} \qquad \text{---(3)}$$

$$1_3 x+m_3 y+n_3 z = \pm\sqrt{\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}} \qquad \text{---(4)}$$

Where li,mi,ni, i = 1,2,3 are d.c's of the normals to these planes

$$\therefore \sum l_1^2 = 1; \quad \sum m_1^2 = 1; \quad \sum n_1^2 = 1;$$

$$\sum l_1 l_2 = 0; \quad \sum 1_1 m_2 = 0$$

$l_1^2+m_1^2+n_1^2=1$ etc.

$\therefore$ By eliminating 1_l, m_1, n_1; $1_2, m_2, n_2$,; 1_3, m_3, n_3

Between (2), (3), (4)

We get $x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

Which is the equation of the director sphere of the conicoid $ax^2+by^2+cz^2=1$

14.6. Examples

1. Prove that equation of two tangent planes to the conicoid $ax^2+by^2+cz^2=1$ --(1) which pass through the line $u=lx+my+nz-p=0$,

$$u'=l'x+m'y+n'z-p'=0 \text{ is}$$

$$u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)-2uu'\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)+u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0$$

Solution: Equation of any polar through the given line is $u+\lambda u'=0$ ---(A)

$$(lx+my+nz-p)+\lambda(l'x+m'y+n'z-p')=0$$

$$x(l+\lambda l')+y(m+\lambda m')+z(n+\lambda n')-p-\lambda p'=0$$

$$\therefore x(l+\lambda l')+y(m+\lambda m')+z(n+\lambda n') = p+\lambda p' \qquad \text{---(2)}$$

This is of the for Lx+My+Nz =P

$$\therefore L=l+\lambda l';\quad M=m+\lambda m';\ N=n+\lambda n';\quad P=p+\lambda p'$$

The plane (2) touches the coni coid (1)

$$\therefore \frac{L^2}{a}+\frac{M^2}{b}+\frac{N^2}{c}=P^2$$

$$\frac{1}{a}(l+\lambda l')^2+\frac{1}{b}(m+\lambda m')^2+\frac{1}{c}(n+\lambda n')^2 = p+\lambda p' \qquad \text{---(B)}$$

$\therefore$ These are two roots

Hence there are two tangent planes to the conicoid (1)

Eliminate λ between (A) and (B)

$$\frac{(u'l-ul')^2}{a}+\frac{(u'm-um')^2}{b}+\frac{(u'n-un')^2}{c}=(u'p-up')^2$$

ie $u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)-2uu'\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)+u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0$

which is the required equation.

2. Find the equations of two tangent plane to the coincoid $2x^2-6y^2+3z^2=5$ which pass through the line x + 9y – 3z = 0, 3x – 3y + 6z – 5 =0 (April 2006, Bharathiyar, November 2005, April 2004)

Proof: Equation of the conicoid is $2x^2-6y^2+3z^2=5$

$\div 5;$

$$\frac{2}{5}x^2-\frac{6}{5}y^2+\frac{3}{5}z^2=1$$

$$\therefore\ a=\frac{2}{5};\ b=-\frac{6}{5};\ \ c=\frac{3}{5}$$

Equation of any plane through the given line is

$$x+9y-3z+\lambda(3x-3y+6z-5)=0$$

$$x(1+3\lambda)+y(9-3\lambda)+z(6\lambda-3)-5\lambda=0$$

$$x(1+3\lambda)+y(9-3\lambda)+z(6\lambda-3)-5\lambda \qquad \text{---(1)}$$

This is of the form $lx+my+nz=p$

$\therefore \quad l=1+3\lambda; \quad m=9-3\lambda; \quad n=6\lambda-3; \quad P=5\lambda$

Formula

$$\therefore P^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}$$

$$25\lambda=\frac{5}{2}(1+3\lambda)^2-\frac{5}{6}(9-3\lambda)^2+\frac{5}{3}(6\lambda-3)^2$$

On simplification, $\lambda=\pm 1$

Using in (1), the two tangent planes are

$4x + 6y + 3z – 5 = 0; 2x – 12y + 9z – 5 = 0$

14.7. Let us sum up

So far we have studied the new concept of the conicord, tangent plane, enveloping cone and director sphere.

14.8. Check your progress

1) What is the director sphere of the conicord $2x^2$-by^2+$3y^2$=5

2) Write down the equation of the tangent plant at (1,1,1) to the conicord $x^2+y^2-z^2=1$

14.9. Lesson End Activities

1. Find the equations of the tangent planes to the conicoid $x^2+y^2+4z^2=1$ which intersect the line 12x-3y-z=0, z = 1 (April 2006 Bharathiyar)
2. Find the equations of two tangent planes to the conicoid $2x^2+2y^2+y^2=2$ which pass through the line z = 0, x + y =0. (April 2005, Bharathiyar)
3. Find the equations of the tangent plane to the coincoid $7x^2+5y^2+3z^2=60$ which pass through the line 7x + 10y – 30 = 0, 5y – 3z = 0 (April 2005, November 2004,Bharathiyar)
4. Find the point of contact of the plane x + 2y + z – 2 = 0 which touches the conicoid $x^2+2y^2-z^2-2=0$ (April 2004, Bharathiyar)

14.10. Points for discussion

a. Find the equations of two planes that can be drawn through the line x = 4, 3y + 4z = 0 to touch the conicoid $x^2 + 3y^2 - 6z^2 = 4$.

b. (Ans: x + 9y + 12z = 4, x – 9y – 12z = 4)

c. Show that the plane 9x + 8y – 5z = 38 touches the conicoid $3x^2 + 4y^2 - 5z^2 = 38$ and find the point of contact (November 2002 Bharathiar)

d. Find the equations of the tangent planes to $x^2 + y^2 + 4z^2 = 1$ which intersect in the line whose equations are 12x – 3y – 5 = 0, z = 1. (November 2005 Bharathiar).

14.11 References

Analytical Geometry of Three Dimension by N.P. Bali