

A TEXT BOOK OF CLASSICAL ALGEBRA AND CALCULUS

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UNIT 1

Lesson – 1

BINOMIAL THEOREMS

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1.0 AIMS AND OBJECTIVES

Our Aim is to learn about the expansion of a Binomial Theorem for a rational index using vandermonde’s theorem. Further we aim at learning problems to be solved using the different types in Binomial series.

1.1 BINOMIAL THEOREM

1.1.1 Vandermonde’s theorem

If n is a positive integer, then $(p+q)^n = p^n + nC_1 p^{n-1}q + nC_2 p^{n-2}q^2 + nC_3 p^{n-3}q^3 + \dots + q^n$ for all values of p and q .

Proof.

If p and q are positive integers, we have

$$(1+x)^p = 1 + \frac{p}{1}x + \frac{p(p-1)}{2!}x^2 + \dots + \frac{p(p-1)\dots(p-r+1)}{r!}x^r + \dots \quad (1)$$

$$(1+x)^q = 1 + \frac{q}{1}x + \frac{q(q-1)}{2!}x^2 + \dots + \frac{q(q-1)\dots(q-r+1)}{r!}x^r + \dots \quad (2)$$

Where $p_r = p(p-1)(p-2)\dots(p-r+1)$

$$q_r = q(q-1)(q-2)\dots (q - \overline{r-1})$$

The coefficient of x^n in the product of two series (1) and (2) is

$$\begin{aligned}
 &= \frac{p_n}{n!} + \frac{-2p_1q_1}{(1-n)!n!} + \frac{2!}{(2n)!} + \dots + \frac{q}{(n!)!} \\
 &= \frac{1}{n!} \left[p_n + \frac{n!}{(-n)!} + \frac{n!}{2!} + \dots + \frac{n!}{3!} + \dots \right] \\
 &= \frac{1}{n!} \left[\binom{n}{0} p_n + \binom{n}{1} (-2) p_1 q_1 + \binom{n}{2} \frac{2!}{2!} + \dots + \binom{n}{n} q \right] \\
 &= \frac{1}{n!} \left[p_n + \frac{n!}{(-n)!} + \frac{n!}{2!} + \dots + \frac{n!}{3!} + \dots \right] \\
 &= \frac{1}{n!} \left[p_n + \frac{n!}{(-n)!} + \frac{n!}{2!} + \dots + \frac{n!}{3!} + \dots \right] \\
 &= \frac{1}{n!} \left[p_n + \frac{n!}{(-n)!} + \frac{n!}{2!} + \dots + \frac{n!}{3!} + \dots \right] \\
 &= \text{Coefficient of } x^n \text{ in the expansion of } (1+x)^{p+q} \\
 &= \frac{(p+q)_n}{n!}
 \end{aligned}$$

$$\therefore (p+q)_n = p_n + nC_1 p_{n-1} q_1 + nC_2 p_{n-2} q_2 + \dots + q_n$$

Book Work 2

1.1.2 Binomial Theorem for a rational Index

Statement

If n is a rational number and $-1 < x < 1$ (i.e) $|x| < 1$, the sum of the series

$$1 + nx + \frac{n(-1)n}{2} x^2 + \frac{(n-1)n}{3} x^3 + \dots + \frac{(n-1)n}{r} x^r + \dots$$

is the real positive value of $(1+x)^n$

Pf: Let $f(x) = 1 + \frac{n_1}{1} x + \frac{n_2}{2!} x^2 + \frac{n_3}{3!} x^3 + \dots + \frac{n_r}{r!} x^r + \dots$

Where $n_r = n(n-1)(n-2)\dots (n - \overline{r-1})$

We denote this series by $u_1 + u_2 + u_3 + \dots + \infty$

$$\frac{u_{r+1}}{u_r} = \frac{\frac{n_r}{r!} x^r}{\frac{n_{r-1}}{(r-1)!} x^{r-1}}$$

$$= \frac{n_r (r-1)! n!}{r! n! r x^{-1}} \therefore n! = n(n-1)!$$

$$= \frac{n_r}{n_{r-1}} \cdot \frac{1}{r} x$$

$$= \frac{n-r+1}{r} x \quad \therefore \frac{n_r}{n_{r-1}} = n-r+1$$

$$\therefore \frac{u_{r+1}}{u_r} = \frac{n-r+1}{r} x$$

$$\therefore \left| \frac{u_{r+1}}{u_r} \right| = \left| \frac{n-r+1}{r} x \right|$$

$$= \left| \frac{n+1}{n-r} - 1 \right| |n|$$

$$\therefore \lim_{x \rightarrow \infty} \left| \frac{u_{r+1}}{u_r} \right| = \lim_{r \rightarrow \infty} \left| \frac{n+1}{n-r} - 1 \right| |n|$$

$$= |-1| |x| \rightarrow r \rightarrow \infty \frac{1}{r}$$

$$= |x|$$

$$\therefore r \rightarrow \infty \left| \frac{u_{r+1}}{u_r} \right| < 1$$

\therefore The series $|u_1| + |u_2| + \dots + |u_r| + \dots + \infty$ is convergent if $|x| < 1$

ie. the series $u_1 + u_2 + u_3 + \dots + u_4 + \dots$ is absolutely convergent.

similarly the series

$$f(x) = p \frac{x^1}{1!} + \frac{p}{2} x^2 + \frac{p}{3!} x^3 + \dots + \frac{p}{n!} x^n + \dots$$

$$f(x) = q \frac{x^1}{1!} + \frac{q}{2} x^2 + \frac{q}{3!} x^3 + \dots + \frac{q}{n!} x^n + \dots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{(p)_n}{n!} x^n \cdot \sum_{n=0}^{\infty} \frac{(q)_n}{n!} x^n = \sum_{n=0}^{\infty} \frac{(p+q)_n}{n!} x^n$$

are also absolutely convergent if $|x| < 1$ $\therefore f(p) + f(q)$ can be multiplied term by term and the resulting series is also absolutely convergent

\therefore The coefficient of x^n in the product of $f(p)$ and $f(q)$ is

$$\frac{(p)_n}{n!} + \frac{(p)_{n-1}(q)_1}{n!(n-1)!} + \frac{(p)_{n-2}(q)_2}{n!(n-2)!} + \dots + \frac{(p)_1(q)_{n-1}}{n!1!} + \frac{(q)_n}{n!}$$

$$= \frac{1}{n!} [p_n + n \binom{p-1}{1} q_1 + \binom{p-2}{2} q_2 + \dots + \binom{p-1}{n-1} q_{n-1} + q_n]$$

By vandermonde's theorem this expression is $\frac{(p+q)_n}{n!}$ and this is the coefficient of $f(p+q)$

\therefore The coefficient of any power of x in $f(p+q)$

= The coefficient of the same power of x in the product $f(p)f(q)$

$\therefore f(p)f(q) = f(p+q)$ for all values of p and q , provides $|x| < 1$.

$\therefore f(p)f(q)f(r) = f(p+q+r)$

$\therefore f(p)f(q)f(r) \dots s \text{ factors} = f(p+q+r \dots s \text{ factors})$

case (1): n is a positive fraction

Let $n = \frac{u}{v}$ where u and v are positive integers.

$f(x) = (1+x)^n$ since u is a positive integer

$$f(0) = 1 \qquad f(1) = 1+x$$

$$f\left(\frac{u}{v}\right) = \left(1 + \frac{x}{v}\right)^u = \left(\frac{v+x}{v}\right)^u = \frac{(v+x)^u}{v^u}$$

$$\left[f\left(\frac{u}{v}\right)\right]^v = \frac{(v+x)^u}{v^u}$$

$$= f(u)$$

$$\left[f\left(\frac{u}{v}\right) \right]^v = (1+x)^v$$

$$f\left(\frac{u}{v}\right) = (1+x)^{u/v}$$

$$\therefore f(x) = (1+x)^u$$

This is the proof for when is a positive fraction.

Case(ii) let n be a negative rational number

Let n=-u where u is a +ve rational number $\therefore n+u=0$

$$f(n) = f(-u)$$

$$= f\left(\frac{1}{(1+x)^u}\right) = \frac{1}{f(x)^u}$$

$$= \frac{1}{(1+x)^u}$$

$$= (1+x)^n$$

$$f(x) = (1+x)^n$$

\therefore The theorem for any negative index

Important Formulas

$$1. (1-x)^{-1} = 1+x+x^2+x^3+\dots\infty$$

$$2. (1+x)^{-1} = 1-x+x^2-x^3+\dots\infty$$

$$3. (1-x)^{-2} = 1+2x+3x^2+4x^3+\dots\infty$$

$$4. (1+x)^{-2} = 1-2x+3x^2-4x^3+\dots\infty$$

1.1.3. Applications of Binomial series

More Formulas

(1) If n is a rational member, and $-1 < x < 1$ ie $|x| < 1$, then

$$(1+x)^n = n \frac{x}{1} + \frac{n(n-1)}{2} \frac{x^2}{1 \cdot 2} + \frac{n(n-1)(n-2)}{6} \frac{x^3}{1 \cdot 2 \cdot 3} + \dots$$

$$2. (1-x)^n = n \frac{-x}{1} + \frac{n(n-1)}{2} \frac{x^2}{1 \cdot 2} + \frac{n(n-1)(n-2)}{6} \frac{x^3}{1 \cdot 2 \cdot 3} + \dots$$

$$3. (1+x)^{-n} = n \frac{x}{1} + \frac{n(n+1)}{2} \frac{x^2}{1 \cdot 2} + \frac{n(n+1)(n+2)}{6} \frac{x^3}{1 \cdot 2 \cdot 3} + \dots$$

$$4. (-x)^{-n} = n! \frac{n(n-1)\dots 2 \cdot 1}{1 \cdot 2 \cdot 3 \dots n} x^{-n} \dots \infty$$

$$5. (-x)^{-p/q} = \frac{p}{1} \left(\frac{p}{q} \right) \frac{p(p+q)}{2} \left(\frac{p+q}{3} \right) \dots x^{-p/q} \dots \infty$$

$$6. (1+x)^{-p/q} = \frac{p}{1} \left(\frac{p}{q} \right) \frac{p(p+q)}{2} \left(\frac{p+q}{3} \right) \dots x^{-p/q} \dots \infty$$

Model 1

1.2. Examples

1. Find the sum to infinity of the series

$$1 + \frac{1}{6} + \frac{1 \cdot 4}{6 \cdot 12} + \frac{1 \cdot 4 \cdot 7}{6 \cdot 12 \cdot 18} + \dots \infty$$

$$s = 1 + \frac{1}{6} + \frac{1 \cdot 4}{6 \cdot 12} + \frac{1 \cdot 4 \cdot 7}{6 \cdot 12 \cdot 18} + \dots \infty$$

$$= 1 + \frac{1}{6} + \frac{1 \cdot 4}{6 \cdot 12} + \frac{1 \cdot 4 \cdot 7}{6 \cdot 12 \cdot 18} + \dots \infty$$

$$= 1 + \frac{1}{6} + \frac{1 \cdot 4}{6 \cdot 12} + \frac{1 \cdot 4 \cdot 7}{6 \cdot 12 \cdot 18} + \dots \infty$$

$$= 1 + \frac{1}{6} + \frac{4}{2 \cdot 6} + \frac{4 \cdot 7}{3 \cdot 6} + \dots \infty$$

$$= 1 + \frac{1}{1} + \frac{4}{2} + \frac{4 \cdot 7}{3} + \dots \infty$$

Which is of the form

$$= 1 + \frac{p}{1} \left(\frac{p}{q} \right) \frac{p(p+q)}{2} \left(\frac{p+q}{3} \right) \dots x^{-p/q} \dots \infty$$

$$\therefore p=1 \quad p+q=4 \quad \frac{x}{q} = \frac{1}{6}$$

$$1+q=4 \quad \frac{x}{3} = \frac{1}{6}$$

$$q=3 \quad x = \frac{3}{6} = \frac{1}{2}$$

$$\therefore s = (1-x)^{-p/q} = (1-1/2)^{-1/3} = \left(\frac{1}{2}\right)^{-1/3}$$

$$= \frac{1}{2^{-1/3}} 2^{1/3}$$

$$= 3\sqrt[3]{2}$$

2. Sum the series

$$\frac{2}{6} + \frac{2 \cdot 5}{6 \cdot 12} + \frac{2 \cdot 5 \cdot 8}{6 \cdot 12 \cdot 18} + \dots \dots \dots \infty \dots$$

$$s = \frac{2}{6} + \frac{2 \cdot 5}{6 \cdot 12} + \frac{2 \cdot 5 \cdot 8}{6 \cdot 12 \cdot 18} + \dots \dots \dots \infty \dots$$

$$= \frac{2}{6 \cdot 1} + \frac{2 \cdot 5}{6 \cdot 2 \cdot 6} + \frac{2 \cdot 5 \cdot 8}{6 \cdot 2 \cdot 6 \cdot 3 \cdot 6} + \dots \dots \dots \infty \dots$$

$$= \frac{2}{6} \cdot \frac{2}{1} \cdot \frac{5}{2} \cdot \frac{1}{6} + \frac{5}{2} \cdot \frac{8}{3} \cdot \frac{1}{6} + \dots \dots \dots \infty \dots$$

$$= \frac{2}{6} \cdot \frac{2}{2} \cdot \frac{5}{1} \cdot \frac{1}{6} + \frac{5}{3} \cdot \frac{8}{6} \cdot \frac{1}{6} + \dots \dots \dots \infty \dots$$

$$= \frac{2}{1} \cdot \frac{1}{1!} \cdot \frac{2}{6} + \frac{5}{2} \cdot \frac{1}{2!} \cdot \frac{2}{6} + \frac{5}{3} \cdot \frac{8}{3!} \cdot \frac{1}{6} + \dots \dots \dots \infty \dots$$

Adding 1 both sides

$$1+s = 1 + \frac{2}{1} \cdot \frac{1}{1!} \cdot \frac{2}{6} + \frac{5}{2} \cdot \frac{1}{2!} \cdot \frac{2}{6} + \frac{5}{3} \cdot \frac{8}{3!} \cdot \frac{1}{6} + \dots \dots \dots \infty \dots$$

$$\therefore p=2 \quad p+q=5 \quad \frac{x}{q} = \frac{1}{6}$$

$$1+q=5 \quad \frac{x}{3} = \frac{1}{6}$$

$$q=3 \quad x = \frac{3}{6} = \frac{1}{2}$$

$$\therefore 1+s = (1-x)^{-p/q}$$

$$= (1-1/2)^{-2/3} = (1/2)^{-2/3} = 2^{2/3}$$

$$\therefore 1+s = 2^{2/3} \therefore s = 2^{2/3} - 1$$

3. Sum the series

$$3 + \frac{3 \cdot 5}{1 \cdot 2} + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} + \frac{7}{3} + \dots \dots \dots \alpha$$

$$s = 3 \frac{3 \cdot 5 \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 3 \cdot 3} \dots \alpha$$

$$= 3 \frac{3 \cdot (5 \cdot 1) \cdot 3 \cdot (5 \cdot 7) \cdot 1}{2 \cdot (3 \cdot 1) \cdot 2 \cdot (3 \cdot 3) \cdot 3} \dots \alpha$$

Multiply both sides by 1/3

$$\frac{1}{3} s = \frac{3}{3} \cdot \frac{5 \cdot 1}{2 \cdot (3 \cdot 1)} \cdot \frac{5 \cdot (7 \cdot 1)}{3 \cdot (3 \cdot 3)} \dots \alpha$$

$$\therefore p=3 \qquad p+q=5 \qquad \frac{x}{q} = \frac{1}{3}$$

$$q=2 \qquad \frac{x}{2} = \frac{1}{3}$$

$$x = \frac{2}{3}$$

$$\therefore 1 + \frac{1}{3} s = (1-x)^{-p/q}$$

$$= (1 - 2/3)^{-3/2} = \left(\frac{1}{3}\right)^{3/2} = 3\sqrt{3}$$

$$\therefore 1 + \frac{1}{3} s = 3\sqrt{3} - 1$$

$$\therefore s = 3[\sqrt{3} - 1]$$

4. Sum the series to infinity

$$\frac{3}{2 \cdot 4} + \frac{3 \cdot 4}{2 \cdot 4 \cdot 6} + \frac{3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8} \dots \infty \dots$$

$$\text{let } S = \frac{3}{2 \cdot 4} + \frac{3 \cdot 4}{2 \cdot 4 \cdot 6} + \frac{3 \cdot 4 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \dots \infty \dots$$

$$= \frac{3}{1 \cdot 2 \cdot 2} + \frac{3 \cdot 4}{1 \cdot 2 \cdot 2 \cdot 2 \cdot 3} + \frac{4 \cdot 5}{1 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 2 \cdot 4} \dots \dots \dots$$

$$s = \frac{3}{1 \cdot (2 \cdot 2 \cdot 1)} + \frac{4}{2 \cdot (3 \cdot 2 \cdot 1)} + \frac{4 \cdot (5)}{2 \cdot (3 \cdot 4 \cdot 2)} \dots \dots \dots$$

Inserting the missing factors in the numerator

Multiply both sides by 2

[3,4,5 are in Ap
the preceding term is 2]

$$\begin{aligned}
 s &= 1 \frac{1}{6} \frac{1}{6} \frac{3}{12} \frac{1}{6} \frac{3}{12} \frac{5}{18} \dots \alpha \\
 &= 1 \frac{1}{6 \cdot 6 \cdot 2 \cdot 6} \frac{3}{1 \cdot 6 \cdot 2 \cdot 6 \cdot 3 \cdot 6} \dots \alpha \\
 &= 1 \frac{1}{6 \cdot 1 \cdot 6 \cdot 2} \frac{3 \binom{1+2}{1}}{(6 \cdot 6 \cdot 1)} \frac{3 \binom{3+5}{3}}{(6 \cdot 3)} \frac{1}{6 \cdot 6} \dots \alpha \\
 &= 1 \frac{1}{1} \frac{\binom{1}{1}}{\binom{1}{1} \binom{1}{6}} \frac{3}{2} \frac{\binom{1}{1}}{\binom{1}{1} \binom{1}{6}} \frac{3}{3} \frac{\binom{3+5}{3}}{\binom{3}{3} \binom{5}{6}} \frac{1}{6} \dots \alpha \\
 &= 1 \frac{p}{1} \frac{(x/p)}{1} \frac{q}{2} \frac{(x/q)}{1} \frac{(x)}{6} \dots \alpha
 \end{aligned}$$

$$\begin{aligned}
 \therefore p=1 & & p+q=3 & & \frac{x}{q} = \frac{1}{6} \\
 & & q=2 & & \frac{x}{2} = \frac{1}{6} \\
 & & & & x = \frac{2}{6} \\
 & & & & x = \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \therefore s &= (1 - x)^{-p/q} \\
 &= \left(1 - \frac{1}{3}\right)^{-1/2} = \left(\frac{2}{3}\right)^{-1/2} = \sqrt{\frac{3}{2}} \\
 &= \sqrt{3}/2
 \end{aligned}$$

(2) Prove that

$$\begin{aligned}
 &\frac{1}{2} \frac{1}{4} \frac{3}{2} \frac{1}{2} \frac{3}{4} \frac{5}{2} \dots \alpha \\
 &= \frac{1}{3} [1 \ 7 \ 9 \ 8] \sqrt{0} \dots
 \end{aligned}$$

Solution

$$\begin{aligned}
 l e \neq s &= \frac{1}{2} \frac{1}{4} \frac{3}{2} \frac{1}{2} \frac{3}{4} \frac{5}{2} \dots \alpha \\
 &= \frac{1}{3 \cdot 8} \frac{1}{3 \cdot 8 \cdot 4} \frac{3}{8 \cdot 3 \cdot 8 \cdot 4} \frac{1}{8 \cdot 3 \cdot 8 \cdot 4 \cdot 8 \cdot 5 \cdot 8} \dots \alpha \\
 s &= \frac{1}{3} \frac{\binom{1}{1}}{\binom{1}{8} \binom{1}{3}} \frac{3}{4} \frac{\binom{1}{1}}{\binom{1}{8} \binom{1}{3}} \frac{3}{4} \frac{\binom{3+5}{3}}{\binom{3}{5} \binom{5}{8}} \frac{1}{8} \dots \alpha.
 \end{aligned}$$

Multiply both sides by $\frac{1}{2}$

$$\frac{1}{2} s = \frac{1}{3} \left(\frac{1}{8} \right)^3 - \frac{1}{4} \left(\frac{1}{8} \right)^4 + \frac{3}{5} \left(\frac{1}{8} \right)^5 - \dots \alpha.$$

$$\frac{1}{2} s = \frac{1}{3} \left(\frac{1}{8} \right)^3 - \frac{1}{4} \left(\frac{1}{8} \right)^4 + \frac{3}{5} \left(\frac{1}{8} \right)^5 - \dots \alpha.$$

Multiply both sides by $\left(\frac{1}{8}\right)^3$

$$\frac{1}{2} s \frac{1}{6} = \frac{1}{4} \left(\frac{1}{8} \right)^3 - \frac{1}{4} \left(\frac{1}{8} \right)^4 + \frac{3}{5} \left(\frac{1}{8} \right)^5 - \dots \alpha.$$

Multiply both sides by negative sign in order to suit formula $(1+x)^{-p/q}$

$$\therefore \frac{s}{12} = \frac{1}{3} \left(\frac{1}{8} \right)^3 - \frac{1}{4} \left(\frac{1}{8} \right)^4 + \frac{1}{5} \left(\frac{1}{8} \right)^5 - \dots \alpha$$

Nr. 1,3,5,...is an AP

common different = 2 the proceeding factor of $1=1-2=-1$

proceeding factor of $-1=-1-2= -3$

\therefore Multiply both sides by (-1)

$$\therefore \frac{s}{12} = \frac{(-1)}{3} \left(\frac{1}{8} \right)^3 - \frac{1}{4} \left(\frac{1}{8} \right)^4 + \frac{1}{5} \left(\frac{1}{8} \right)^5 - \dots \alpha$$

Multiply both sides by -3

$$\frac{-3}{12} s = \frac{(-1)}{3} \left(\frac{1}{8} \right)^3 - \frac{1}{4} \left(\frac{1}{8} \right)^4 + \frac{1}{5} \left(\frac{1}{8} \right)^5 - \dots \alpha$$

Inserting the missing terms

Add $\frac{(-3)}{2} \left(\frac{1}{8} \right)^3$ on both sides

$$\frac{-3}{12} s + \frac{(-3)}{8} \left(\frac{1}{8} \right)^3 = \frac{(-1)}{3} \left(\frac{1}{8} \right)^3 - \frac{1}{4} \left(\frac{1}{8} \right)^4 + \frac{1}{5} \left(\frac{1}{8} \right)^5 - \dots \alpha$$

Add 1 both sides

$$-\frac{3}{12} s + \frac{1}{8} \left(\frac{1}{8} \right)^3 = \frac{1}{3} \left(\frac{1}{8} \right)^3 - \frac{1}{4} \left(\frac{1}{8} \right)^4 + \frac{1}{5} \left(\frac{1}{8} \right)^5 - \dots \alpha$$

$$\frac{-3}{1} s + \frac{3}{2} \frac{x}{8} = \frac{1}{2} q \frac{x}{8} \left(\frac{p}{1} \right) \frac{p}{1} + \frac{q}{2} \left(\frac{p}{1} \right) \frac{p}{2} \frac{q}{1} \left(\frac{p}{1} \right) \dots \alpha$$

$$\frac{-3}{1} s + \frac{3}{2} \frac{x}{8} = \frac{1}{2} q \frac{x}{8} (1 - x)^{1/q}$$

$$\frac{-3}{1} s + \frac{7}{2} \frac{x}{8} = \frac{9}{8} (1 - x)^{1/3}$$

$$p = -3; p + q = -1 \quad x/q = 1/8$$

$$= \left(\frac{5}{4} \right)^{3/2}$$

$$-3 + q = -1 \quad x/2 = 1/8$$

$$= \frac{5\sqrt{5}}{(3/2)^2} \frac{5\sqrt{5}}{8}$$

$$q = 2 \quad x = 2/8 =$$

1/4

$$\frac{-3}{1} s = \frac{5}{8} \frac{5\sqrt{1}}{8} \frac{7}{1} \frac{1}{2}$$

$$\frac{-3}{1} s = \frac{8}{8} \frac{0 - \sqrt{1}}{1} \frac{7}{2} \frac{1}{8}$$

$$3s = 7 \frac{9}{8} \sqrt{0}$$

$$s = \frac{1}{3} (1 - 7 \frac{9}{8} \sqrt{0})$$

1.3 Let us sum up

We have seen the expansions of $(1+x)^{-n}$, $(1-x)^{-n}$, $(1+x)^n$, $(1-x)^n$, $(1-x)^{-p/q}$, $(1+x)^{-p/q}$ and the sum to infinity of a given Binominal series in any one of the above form.

1.4. Check you progress

(1) Write down the expansions of the following

- (1) $(2x-y)^{1/2}$ (2) $(1-2y)^{-3/2}$ (3) $(1+5x)^{7/2}$

1.5 Lesson End Activities

Binominal Series

Exercise

Find the sum of the infinite series.

1) $1 + \frac{3}{2^3} + \frac{1}{1} \cdot \frac{3^2}{2^6} + \frac{3}{2} \cdot \frac{3}{2} \cdot \frac{3^3}{2^9} + \dots \alpha$ (A 2 s

2) $1 + \frac{4}{6} + \frac{4}{6} \cdot \frac{5}{9} + \frac{4}{6} \cdot \frac{5}{9} \cdot \frac{6}{12} \dots \alpha$ $\left(A \frac{1}{n} \right)^9$

$$3) 1 - \frac{3}{4} + \frac{3}{8} - \frac{5}{4} + \frac{5}{8} - \frac{7}{12} \cdot \alpha \quad \text{Ans} \left(\frac{3}{2} \right)^{3/2}$$

$$4) 1 + \frac{1}{4} + \frac{1}{8} + \frac{4}{8} + \frac{1}{8} + \frac{4}{8} + \frac{7}{12} + \frac{1}{4} + \frac{7}{8} + \frac{1}{12} + \frac{1}{6} \cdot \alpha \quad (\sqrt[3]{4})$$

$$5) 1 + \frac{5}{8} + \frac{5}{8} + \frac{8}{12} + \frac{5}{8} + \frac{8}{12} + \frac{1}{6} \cdot \alpha \quad (4 \times 2^{1/3} - 2)$$

1.6. Points for discussion

6) Prove that

$$1 + \frac{1}{4} + \frac{1}{8} + \frac{3}{4} + \frac{1}{8} + \frac{3}{8} + \frac{5}{12} \cdot \alpha$$

$$= \left(1 + \frac{1}{3} + \frac{1}{3} + \frac{3}{6} + \frac{1}{3} \right) \frac{3}{6} + \frac{5}{9} \cdot \alpha \quad \times \quad \left(1 - \frac{1}{4} + \frac{1}{8} + \frac{3}{4} - \frac{1}{8} + \frac{3}{8} - \frac{5}{12} \right) \cdot \alpha$$

7) Prove that

$$1 + n \frac{2n}{1+n} \left(\frac{n-1+n}{2n} \right) \left(\frac{2n}{1} \right)^2 \cdot \alpha$$

$$= 1 + n \frac{2n}{1-n} \left(\frac{n-1-n}{2n} \right) \left(\frac{2n}{1} \right)^2 \cdot \alpha$$

8) Prove that, if n is a positive integer

$$1 - n \frac{1+x}{1-nx^2} + \frac{n(n-2)x(-1n)}{n(n+1)x^2} + \frac{(-1n)(-n2)}{3n!} + \frac{n(1-x^3)}{3(x-1)} \cdot \alpha > 0$$

9) If $x > -1/2$, Prove that

$$\frac{x}{\sqrt{1+x}} = \frac{x}{1+x} + \frac{1}{1} \left(\frac{1}{2+x} \right)^2 + \frac{3}{4} \left(\frac{1-x^3}{2x} \right)^2 + \frac{x^4}{16} + \dots \cdot \alpha$$

10) Prove that

$$\frac{1}{3} + \frac{3}{6} + \frac{1}{6} + \frac{3}{9} + \frac{5}{9} + \frac{1}{9} + \frac{3}{9} + \frac{5}{12} + \frac{7}{12} \cdot \alpha \dots 0 \text{ a p q}$$

1.7. References

A Text Book of Algebra by T.K. Manickavasagam Pillai

Lesson - 2

EXPONENTIAL SERIES

Contents

- 2.0. Aims and Objectives
- 2.1. The Exponential Theorem or Exponential series for all real values of x
- 2.2. Example
- 2.3. Let us sum up
- 2.4. Check your progress
- 2.5. Lesson End Activities
- 2.6. Points for discussion
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2.0. Aims and Objectives

Our aim is to study use expansions of e^x , e^{-x} , and problems to find the sum to infinity of a given series.

Exponential Series

2.1. The Exponential Theorem or Exponential series for all real values of x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^n}{n!} + \dots$$

proof: If n is a +ve integer,

$$\begin{aligned} \left(1 + \frac{x}{n}\right)^n &= 1 + n \cdot \frac{x}{n} + \frac{n(n-1)}{2!} \left(\frac{x}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{x}{n}\right)^3 + \dots + \frac{n(n-1)\dots(n-r+1)}{r!} \left(\frac{x}{n}\right)^r + \dots \\ &= 1 + x + \frac{x^2}{2!} p_1 + \frac{x^3}{3!} p_2 + \dots + \frac{x^r}{r!} p_{r-1} + \dots \end{aligned}$$

where $p_r = (1-1/n)(1-2/n)\dots(1-r/n)$

Now $p_r < 1$ since $1/n, 2/n, 3/n, \dots, \frac{n-1}{n}$ are +ve numbers less than unity,

for the values of 2, 3, ..., (n-1) of r, we have

$$p_r > 1 - \frac{1}{n} (1 + 2 + 3 + \dots + r) = 1 - \frac{r(r+1)}{2n}$$

$$\therefore 0 < 1 - p_r < \frac{r(r+1)}{2n}$$

Let S_n denote the sum to n terms of the series

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

and let $|x| = x_1$

$$\therefore S_{n+1} = \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}\right) + \frac{x^{n+1}}{(n+1)!}$$

$$< \frac{x_1^2}{2^n} \left[\frac{1}{2} + \frac{2}{2} \cdot \frac{2}{2} + \frac{3}{3} \cdot \frac{3}{2} + \dots + \frac{4}{4} \cdot \frac{4}{2} \dots \right] + \dots$$

$$< \frac{x_1^2}{2^n} \left(\frac{x_1^2}{2} + \frac{x_1^3}{3} + \dots \right) + \dots$$

$$< \frac{x_1^2}{2x} E(x) \quad \text{where } E(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$$

Now $E(x_1)$ is finite for all values of x_1

$$\therefore \lim_{n \rightarrow \infty} S_{n+1} = e^x$$

but $\lim_{n \rightarrow \infty} S_{n+1} = E(x)$

$$e^x = E(x)$$

This result is called Exponential Theorem

More Formulas

$$1) e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$2) e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

$$3) \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

$$4) \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$5) e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$$

$$6) e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots$$

$$7) \frac{e + e^{-1}}{2} = 1 + \frac{1}{4!} + \frac{1}{16!} + \dots$$

$$\begin{aligned}
 &= 1 - e^{-x} = 1 - e^{-1} \left(\frac{1}{2} \right)^{2x} \\
 &= 1 - e^{-1 \left(\frac{1}{2} \right)} \\
 &= 1 - \frac{1}{2} = \frac{1}{2}
 \end{aligned}$$

$\therefore s = 1/2$

3) Show that

$$\begin{aligned}
 &\left[1 + \frac{m}{1} \right] \frac{m^2}{2!} \left(\frac{1}{1!} \left[\frac{1}{1!} \frac{1}{m} \left(\frac{1}{2!} \frac{1}{m} \right)^2 \right] \right) \dots \\
 &= e^{-2} \left[1 - \frac{(m-3)}{1} \frac{m}{m!} + \frac{(m-4)}{2!} \frac{1}{m} \right] \dots \quad (\text{April 1986 Bharathiar Univ.})
 \end{aligned}$$

Solution :

$$\begin{aligned}
 LHS &= \left(1 - \frac{m}{1} + \frac{m^2}{2!} \right) \dots \alpha \left(1 - \frac{1}{1!} \left(\frac{1}{m} \right) + \frac{1}{2!} \left(\frac{1}{m} \right)^2 \right) \dots \\
 &= e^{-m} \cdot \frac{1}{m!} \cdot m^{m+1} \dots (1) \\
 RHS &= e^{-2} \left(1 - \frac{1}{1} + \frac{(m-3)}{2!} \frac{m}{m!} + \frac{(m-4)}{3!} \frac{1}{m} \right)^2 \dots \alpha \\
 &= e^{-2} \cdot e^{-m} \\
 &= e^{-2} \frac{(m-3)}{m} \\
 &= e^{-\frac{2m + m - 3}{m}} \\
 &= e^{-\frac{2m + m^2 + 2m + 1 - 3}{m}} \\
 &= e^{-\frac{m^2 + 1}{m}} \\
 &= e^{-m - \frac{1}{m}} \dots \dots (
 \end{aligned}$$

from (1) and (2)

LHS = RHS

Model 2

1) Find the sum to infinity:

$$\frac{3}{2} + \frac{4}{3} + \frac{5}{4} + \dots \quad (\text{April 1986 Bharathiar})$$

Numerator

Consider the first factors 3,4,5,6,.....

This is an Arithmetic Progression

$$a=3, d=1,$$

$$\begin{aligned} n^{\text{th}} \text{ term} &= a+(n-1)d \\ &= 3+(n-1) \\ &= n+2 \end{aligned}$$

Consider the second factors 4,5,6,7,.....

This is an AP with a=4, d=1

$$n^{\text{th}} \text{ term} = 4+(n-1)=n+3$$

Denominator

Consider 2,3,4,5,.....

This is an AP with a=2, d=1

$$n^{\text{th}} \text{ term} = 0+n-1=n+1$$

Let t_n denote the n^{th} term of the given series

$$\begin{aligned} \therefore t_n &= \frac{(n+2)(n+3)}{(n+1)!} \\ &= \frac{n^2+5n+6}{(n+1)!} \end{aligned}$$

$$\text{Let } n^2+5n+6=A+B(n+1)+c(n+1),n$$

To find A,B,C

Put $n=-1$ in(1)

$$(-1)^2+5(-1)+6=A$$

$$\boxed{A=2}$$

Put $n = 0$ in (1)

$$6 = A+B$$

$$6 = 2+B$$

$$\therefore \boxed{B=4}$$

Compare the coefficient of n^2

$$1=c \quad \therefore \boxed{C=1}$$

$$\therefore t_n = \frac{A + B + h}{(n+1)!} C (n+1)^h$$

$$t_n = \frac{2+4 + h}{(n+1)!} (n+1)^h$$

$$= 2 \frac{1}{(n+1)!} \left(\frac{1}{n+1} + \frac{n+1}{n+1} \right)$$

$$= 2 \frac{1}{(n+1)!} \left(\frac{1}{n+1} + \frac{n+1}{n+1} \right)$$

$$t_n = 2 \left(\frac{1}{(n+1)n!} + \frac{1}{n+1} \right)$$

$$t_1 = 2 \left(\frac{1}{2!} + \frac{1}{1!} \right)$$

$$t_2 = 2 \left(\frac{1}{3!} + \frac{1}{2!} \right)$$

$$t_3 = 2 \left(\frac{1}{4!} + \frac{1}{3!} \right)$$

Adding vertically

$$t_1+t_2+t_3+\dots+\alpha = 2 \left(\frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots + \alpha \right)$$

$$+ 4 \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \alpha \right)$$

$$+ \left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \dots + \alpha \right)$$

$$s = 2 \left(\frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} \right) + \frac{1}{3!} + \frac{1}{4!} + \dots + \left(\frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} \right) + \alpha$$

$$= 2(e-1-1) + 4(e-1) + e$$

$$= 2(e-2) + 4e - 4 + e$$

$$= 2e^{-4} + 4e^{-4} + e^{-4}$$

$$S = 7e^{-4}$$

2) Sum the series

$$1 + \frac{1+3}{2!} + \frac{1+3+3^2}{3!} + \frac{1+3+3^2+3^3}{4!} + \dots \quad \text{(Bharatiar April 2005)}$$

$$s = \frac{1}{1!} + \frac{1+3}{2!} + \frac{1+3+3^2}{3!} + \frac{1+3+3^2+3^3}{4!} + \dots$$

Let t_n denote the n^{th} term of the given series

$$\therefore t_n = \frac{1+3+3^2+\dots+3^{n-1}}{n!}$$

$1+3+3^2+\dots+3^{n-1}$ is a Geometric series with $a=1, r=3 > 1$

$$t_n = \frac{1(3^n - 1)}{3 - 1}$$

$$= \frac{1}{2} \frac{3^n - 1}{n!}$$

$$t_n = \frac{1}{2} \left[\frac{3^n}{n!} - \frac{1}{n!} \right]$$

$$t_1 = \frac{1}{2} \left[\frac{3}{1!} - \frac{1}{1!} \right]$$

$$t_2 = \frac{1}{2} \left[\frac{3^2}{2!} - \frac{1}{2!} \right]$$

$$t_3 = \frac{1}{2} \left[\frac{3^3}{3!} - \frac{1}{3!} \right]$$

Adding

$$S = \frac{1}{2} \left[\frac{3}{1!} - \frac{1}{1!} + \frac{3^2}{2!} - \frac{1}{2!} + \frac{3^3}{3!} - \frac{1}{3!} + \dots \right] = \left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \right) \cdot \alpha$$

$$= \left[\frac{n}{1!} + \frac{n^2}{2!} + \frac{n^3}{3!} + \dots \right] \cdot \alpha = (e^n - 1)$$

$$= \frac{1}{2} [(e^n - 1) - 1] = \frac{1}{2} (e^n - 2) = \frac{1}{2} (e^n - 2)$$

$$= \frac{1}{2} [e^3 - 2] = \frac{1}{2} (e^3 - 2)$$

3) Show that $\frac{5}{1} + \frac{7}{3} + \frac{9}{5} + \frac{11}{7} + \dots = \frac{5}{2} - \frac{3}{2e}$

Solution : Numerator

Consider the numbers 5,7,9,11,.....

This is an AP with a=5, d=2

$$n^{\text{th}} \text{ term} = a+(n-1)d$$

$$= 5+(n-1)2$$

$$= 5+2n-2$$

$$= 2n+3$$

Denominator

Consider the numbers 1,3,5,.....

This is an AP with a=1, d=2

$$n^{\text{th}} \text{ term} = a+(n-1)d$$

$$= 1+(n-1)2$$

$$= 1+2n-2$$

$$= 2n-1$$

Let t_n denote the n^{th} term of the given series

$$\therefore t_n = \frac{(2n+3)}{(2n-1)}$$

Let $2n+3=A+B(2n-1)$

put $n=1$; $5=A+B$

Put $n=1/2$, $4=A$

$$\therefore B=1$$

$$A=4, B=1$$

$$\therefore t_n = \frac{A+B(2n-1)}{(2n-1)}$$

$$= \frac{4+(2n-1)}{(2n-1)}$$

$$= 4 \frac{1}{(2n-1)} + \frac{1}{(2n-1)}$$

$$= 4 \frac{1}{(n-1)!} \frac{(n-1)}{(n-1)(n-2)}$$

$$t_n = 4 \frac{1}{(n-1)!} \frac{1}{n(n-2)}$$

$$t_1 = 4 \frac{1}{1!} \frac{1}{0}$$

$$t_2 = 4 \frac{1}{3!} \frac{1}{2}$$

$$t_3 = 4 \frac{1}{5!} \frac{1}{4}$$

Adding vertically

$$S = t_1 + t_2 + t_3 + \dots$$

$$= 4 \left[\frac{1}{1!} - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \dots \right] \left[\frac{1}{0} - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \dots \right]$$

$$= 4 \left(\frac{e - e^{-1}}{2} \right) \frac{e^{-1} e}{2}$$

$$= \frac{1}{2} [4e - 4e^{-1}] e^{-1} e$$

$$= \frac{1}{2} [5e - 3e]$$

$$= \frac{1}{2} [5 - 3] e$$

$$= \frac{5e}{2} - \frac{3}{2}e$$

4) Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{n^2 + 3}{(n+2)!} x^n$$

$$t_n = \frac{n^2 + 3}{(n+2)!} x^n$$

$$= \frac{(n^2 + 3)(n+1)}{(n+2)!} x^n$$

$$t_n = \frac{(n+1)(n+3)}{(n+2)!} x^n$$

$$\text{let } (n+1)(n^2+3) = A+B(n+2)+C(n+2)(n+1)+D(n+2)(n+1)n \dots (1)$$

Put $n = -2$ in (1), $A = -7$

Put $n = -1$ in (1)

$$0 = A+B$$

$$0 = -7+B$$

$$\therefore B = 7$$

Put $n = 0$ in (1)

$$3 = A+2B+2C$$

$$C = -2$$

Compare the coefficient of n^3 on both side of (1)

$$1 = D$$

$$A = -7; B = 7; C = -2; D = 1$$

Step 2

$$\therefore t_n = \frac{A + B \binom{n}{2} + C \binom{n}{2} \binom{n}{1} + D \binom{n}{2} \binom{n}{1} \binom{n}{0}}{(n-2)!}$$

$$t_n = \frac{A}{(n-2)!} + \frac{B \binom{n}{2}}{(n-2)!} + \frac{C \binom{n}{2} \binom{n}{1}}{(n-2)!} + \frac{D \binom{n}{2} \binom{n}{1} \binom{n}{0}}{(n-2)!}$$

$$= 7 \frac{1}{(n-2)!} + \frac{7 \binom{n}{2}}{(n-2)!} + \frac{-2 \binom{n}{2} \binom{n}{1}}{(n-2)!} + \frac{1 \binom{n}{2} \binom{n}{1} \binom{n}{0}}{(n-2)!}$$

$$t_n = 7 \frac{1}{(n-2)!} + \frac{7}{(n-2)!} + \frac{2}{(n-2)!} + \frac{1}{(n-2)!}$$

$$t_1 = 7 \frac{1}{3!} + \frac{7}{2!} + \frac{2}{1!} + \frac{1}{0!}$$

$$t_2 = 7 \frac{1}{4!} + \frac{7}{3!} + \frac{2}{2!} + \frac{1}{1!}$$

$$t_3 = 7 \frac{1}{5!} + \frac{7}{4!} + \frac{2}{3!} + \frac{1}{2!}$$

Add

$$S = t_1 + t_2 + t_3 + \dots + \alpha$$

$$= 7 \left(\frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{\alpha!} \right) + \frac{1}{2!} + \alpha \left(\frac{1}{3!} + \frac{1}{4!} + \dots + \frac{1}{\alpha!} \right)$$

$$\begin{aligned}
 & -2 \left(\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \dots \right) \cdot \alpha \\
 & = 7 \left(e^{1-7} - \frac{1}{2} e^{1-7} + (1-7) e^{2-7} + \dots \right) \\
 \therefore e &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots \cdot \alpha \\
 & = -e^{-3/2}
 \end{aligned}$$

2.3. Let us sum up

We have studied so far the series e^x and hence e^{-x} . Also the results related to e^x and e^{-x} we have also studied about the sum to infinity of a given series in this lesson.

2.4. Check your progress

(1) What is the value of $1 + \frac{1}{3!} + \frac{1}{5!} + \dots \cdot \alpha$.

(2) Write down the value of the series $\frac{3}{1!} + \frac{3^3}{3!} + \frac{3^5}{5!} + \dots \cdot \alpha$.

Exponential Series

2.5. Lesson End Activities

1) Prove that

$$1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots = \alpha \frac{3}{2}$$

2) Prove that

$$\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \frac{8}{9!} + \dots = \frac{1}{e}$$

3) Prove that

$$1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots = \frac{1}{2} (e^x + e^{-x})$$

4) Prove that

$$\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \dots = e^{-1}$$

5) Prove that

$$\frac{1^2}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \dots = e'$$

6) Prove that

$$1 + \frac{(+1-2+)}{2!} + \frac{(+1-2+3-)}{3!} + \frac{(+1-2+3-4)}{4!} = \frac{e}{2}$$

2.6. Points for discussion

7) Prove that

$$1^3 + \frac{2^3}{1!} + \frac{3^3}{2!} + \frac{4^3}{3!} + \dots = 5e$$

8) Prove that

$$1^3 + \frac{(+2-2)^2}{1!} + \frac{(+2-2+3^2)}{2!} + \frac{(+2-2+3^2-4)}{3!} = \frac{4e}{6}$$

9) Prove that

$$\frac{1}{1!}x + \frac{(1+2)}{2!}x^2 + \frac{(1+2+3)}{3!}x^3 = e^x \cdot \alpha \cdot e^{-x}$$

10) Prove that

$$\frac{1^3 x^3}{1!} + \frac{2^3 x^3}{2!} + \frac{3^3 x^3}{3!} + \frac{4^3 x^4}{4!} = x^3 \cdot e^{x^3+1}$$

11) Prove that

$$\frac{2}{3!} + \frac{3}{4!} + \frac{3}{5!} + \frac{5}{6!} + \frac{7}{6!} + \dots = e^{2\frac{7}{2}}$$

12) Prove that

$$\frac{1^2}{1!} + \frac{2^2}{2!} + \frac{3^2}{3!} + \frac{3^2}{4!} = 2e'$$

13) Prove that

$$\frac{1^3}{1!} + \frac{1+2}{2!} + \frac{1^3+2^3}{3!} + \frac{1^3+2^3+3^3}{4!} = \frac{2}{4} \cdot \frac{7}{4} e$$

14) Prove that

$$\sum \frac{\binom{n}{k} e^{-1}}{(k+1)!} = \frac{1}{2}$$

15) Prove that

$$\sum \frac{\binom{n}{k} (n^k + 2n^{k-1})}{(k+1)!} = 2e - 1$$

2.7. References

A Text Book of Algebra by T.K. Manickavasagam Pillai

Lesson-3

LOGARITHMIC SERIES

Contents

- 3.0 Aims and Objectives
- 3.1. Logarithmic Theorem or logarithmic series
- 3.2. Examples
- 3.3. Let us sum up
- 3.4. Check your progress
- 3.5. Lesson End Activities
- 3.6. Points for Discussion
- 3.7. References

3.0 Aims and Objectives

Our Aim is to study the series of $\log(1+x)$, $\log(1-x)$ etc

Logarithmic Series

3.1. Logarithmic Theorem or logarithmic series

If $-1 < x < 1$, then

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \alpha$$

from the exponential theorem we have

$$a^y = 1 + y \log a + \frac{y^2}{2} (\log a)^2 + \frac{y^3}{3} (\log a)^3 + \dots \alpha$$

put $a = 1+x$

$$\therefore (1+x)^y = 1 + y \log(1+x) + \frac{y^2}{2} [\log(1+x)]^2 + \dots \alpha$$

But, since $|x| < 1$, we have by Binomial theorem

$$\begin{aligned} (1+x)^y &= 1 + y \left[\frac{x}{1} + \frac{y(y-1)}{2!} x^2 + \frac{y(y-1)(y-2)}{3!} x^3 + \dots \right] \alpha \\ &= 1 + y \left[x + \frac{1 \cdot 2}{2!} x^2 + \frac{1 \cdot 2 \cdot 1}{3!} x^3 + \dots \right] \alpha \end{aligned}$$

$$\therefore 1 + y \log(1+x) + \frac{y^2}{2} [\log(1+x)]^2 + \dots \alpha$$

$$= 1 - y \left(x \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \dots \right)$$

Compare the coefficient of y we get

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

This is known as logarithmic series

FORMULAS

1) When $-1 < x < 1$,

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

2) When $-1 < -x < 1$

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

$$3) \log_e \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

$$4) \log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

3.2. Examples

Model-1

$$1) \text{ Prove that } 1 - \left(\frac{n+1}{n-1} \right)^2 = \frac{2n}{n+1} \left(\frac{1}{3} + \frac{n^2}{n+1} + \frac{1}{5} + \frac{n^4}{n+1} + \dots \right)$$

Proof

$$\text{put } \frac{2n}{n^2+1} = y$$

$$\therefore R = 1 - \left(\frac{n+1}{n-1} \right)^2 = 1 - y^3 + y^5 - \dots$$

$$= \frac{1}{2} \log_e \frac{1+y}{1-y} \quad (\text{formula (3) is used})$$

$$= \frac{1}{2} \log_e \left[\frac{1 + \frac{2n}{n^2+1}}{1 - \frac{2n}{n^2+1}} \right]$$

$$\begin{aligned}
 &= \frac{1}{2} \log \left[\frac{n^2 + 2n + 1}{n^2 - 1} \right] \\
 &= \frac{1}{2} \log \left(\frac{(n+1)^2}{(n-1)(n+1)} \right) \quad \therefore \log \frac{n+1}{n-1} \\
 &= \log \left(\frac{n+1}{n-1} \right) \\
 &= \text{LHS}
 \end{aligned}$$

2) If $-1 < x < 1/3$ prove that

$$2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} \right) \dots \alpha = \frac{2n}{1-n} \frac{1}{2} \left(\frac{n}{1-n} \right)^2 \frac{1}{3} \left(\frac{n}{1-n} \right)^3 \dots \alpha$$

Pf : LHS = $2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} \right) \dots \alpha$

$$= 1 \log \frac{1+x}{1-x} \quad (\text{by formula (3)} \dots (1))$$

RHS put $\frac{2}{1-n} = y$

$$\therefore R = \frac{1}{2} y^2 + \frac{1}{3} y^3 \dots \alpha$$

$$\begin{aligned}
 &= \log(1+y) \\
 &= 1 \log \left[\frac{1+y}{1-y} \right] \\
 &= 1 \log \left[\frac{1+\frac{2}{1-n}}{1-\frac{2}{1-n}} \right] \\
 &= 1 \log \left(\frac{1+n}{1-n} \right) \dots (2)
 \end{aligned}$$

From (1) and (2) LHS = RHS

3) Prove that

$$\frac{3}{1} \left[\frac{1}{0} \frac{1}{2} \frac{1}{2} \frac{3}{2} \frac{1}{4} \right] \frac{3^2}{2^2} \dots \alpha \log \frac{3}{2}$$

solution

$$\begin{aligned}
 &= 1 \circ \left(\frac{1+n}{1-n} \right) \\
 &= 1 \circ \left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}} \right) \\
 &= 1 \circ \left(\frac{\frac{3}{2}}{\frac{1}{2}} \right) \\
 &= 1 \circ \left(\frac{3}{1} \right) \\
 &= \log 3 \\
 &= \text{LHS}
 \end{aligned}$$

5) Prove that

$$\begin{aligned}
 LHS &= \log \left(\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} \cdot \frac{1}{6} \cdot \frac{1}{7} \cdot \frac{1}{8} \right) \quad \alpha \\
 RHS &= \log \left(\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} \cdot \frac{1}{6} \cdot \frac{1}{7} \cdot \frac{1}{8} \right) \quad \alpha \\
 &= \log \frac{1}{2} + \log \frac{1}{3} + \log \frac{1}{4} + \log \frac{1}{5} + \log \frac{1}{6} + \log \frac{1}{7} + \log \frac{1}{8} \quad \alpha \\
 &= \left[\log \frac{1}{3} + \log \frac{1}{4} + \log \frac{1}{5} + \log \frac{1}{6} + \log \frac{1}{7} + \log \frac{1}{8} \right] + \log \frac{1}{2} \quad \alpha \\
 &= \left[\log \frac{1}{3} + \log \frac{1}{4} + \log \frac{1}{5} + \log \frac{1}{6} + \log \frac{1}{7} + \log \frac{1}{8} \right] + \log \frac{1}{2} \quad \alpha \\
 &= 2 \left[\log \frac{1}{3} + \log \frac{1}{4} + \log \frac{1}{5} + \log \frac{1}{6} + \log \frac{1}{7} + \log \frac{1}{8} \right] + \log \frac{1}{2} \quad \alpha \quad \text{where } y = 1/4 \\
 &= 2 \left[\log \left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \frac{1}{8}x^8 \right) \right] + \log \frac{1}{2} \quad \alpha \quad \text{where } x = \frac{1}{2} \\
 &= 1 \circ \frac{1+n}{1-n} \cdot \frac{1}{2} \circ \log (y) \\
 &= 1 \circ \left(\frac{1+\frac{1}{2}}{1-\frac{1}{2}} \right) \cdot \frac{1}{2} \circ \log \left(\frac{1}{4} \right) \\
 &= 1 \circ \frac{3}{1} \cdot \frac{1}{2} \circ \log \frac{1}{4}
 \end{aligned}$$

$$= 1 \circ \sqrt[3]{\frac{1}{4}} \circ \sqrt[3]{\frac{1}{4}}$$

$$= 1 \circ \sqrt[3]{\frac{1}{4}} \circ \sqrt[3]{\frac{1}{4}}$$

$$= 1 \circ \sqrt[3]{\frac{1}{4}} \circ \sqrt[3]{\frac{1}{4}}$$

Model 2

Problems using the formula

$$1 \circ \sqrt[3]{\frac{1}{4}} \circ \sqrt[3]{\frac{1}{4}}$$

1) Prove that

$$1 \circ \sqrt[3]{\frac{1}{4}} \circ \sqrt[3]{\frac{1}{4}}$$

$$R H \circ \sqrt[3]{\frac{1}{4}} \circ \sqrt[3]{\frac{1}{4}}$$

$$\text{The } n^{\text{th}} \text{ term is } t_n = \frac{(-1)^{n+1}}{n(n+1)}$$

splitting $\frac{1}{n(n+1)}$ into partial fractions

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1) + Bn}{n(n+1)}$$

$$\therefore 1 = A(n+1) + Bn$$

$$\text{put } n = -1, B = -1$$

$$\text{put } n = 1 \therefore 1 = A$$

$$\therefore A = 1, B = -1$$

$$\therefore \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\therefore t_n = \left(\frac{1}{n} - \frac{1}{n+1} \right) (-1)^{n+1}$$

$$= \frac{(-1)^{n+1}}{n} - \frac{(-1)^{n+1}}{n+1}$$

$$t_1 = \frac{(-1^2)}{1} \cdot \frac{1}{2}$$

$$t_2 = \frac{1}{2} \cdot \frac{1}{3}$$

$$t_3 = \frac{1}{2} \cdot \frac{1}{3}$$

$$t_4 = \frac{1}{3} \cdot \frac{1}{4}$$

$$t_4 = \frac{1}{4} \cdot \frac{1}{5}$$

$$S = t_1 + t_2 + t_3 + \dots \infty$$

$$= \left(1 - \frac{1}{2}\right) \frac{1}{3} + \frac{1}{4} \left(\frac{1}{2} - \frac{1}{3}\right) + \dots$$

$$= 1 - \log 2 + \log 2 - 1 + \dots$$

$$= \log 2 - 1 + \log 2 - 1$$

$$= \log 2 - 1 + \log 2 - 1$$

$$= 2 \log 2 - \log e$$

$$S = \log(4/e)$$

2) Prove that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \dots$$

let t^n denote the n^{th} term of the given series

$$t_n = \frac{1}{(n+1)n(n-1)}$$

splitting into partial fractions

$$\frac{1}{(n-1)n(n+1)} = \frac{A}{n-1} + \frac{B}{n} + \frac{C}{n+1}$$

$$\frac{1}{(n-1)n(n+1)} = \frac{A(n)(n+1)}{(n-1)n(n+1)} + \frac{B(n-1)(n+1)}{(n-1)n(n+1)} + \frac{C(n-1)n}{(n-1)n(n+1)}$$

$$\therefore 1 = A(n)(n+1) + B(n-1)(n+1) + C(n-1)n$$

by putting $n=0, n=1/2, n=-1/2$

we get $A=1/2, B=-1, C=1/2$

$$\therefore t_n = \frac{1}{2} \cdot \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \frac{1}{2} \cdot \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n+1} - \frac{1}{n+2}$$

$$t_n = \frac{1}{2} \left[\frac{1}{n-1} - \frac{1}{n} + \frac{1}{n+1} - \frac{1}{n+2} \right]$$

$$t_1 = \frac{1}{2} \left[\frac{1}{1-1} - \frac{1}{1} + \frac{1}{1+1} - \frac{1}{1+2} \right]$$

$$t_2 = \frac{1}{2} \left[\frac{1}{2-1} - \frac{1}{2} + \frac{1}{2+1} - \frac{1}{2+2} \right]$$

$$S = t_1 + t_2 + t_3 + \dots \infty$$

$$= \frac{1}{2} \left[\frac{1}{1-1} - \frac{1}{1} + \frac{1}{1+1} - \frac{1}{1+2} \right] + \frac{1}{2} \left[\frac{1}{2-1} - \frac{1}{2} + \frac{1}{2+1} - \frac{1}{2+2} \right] + \dots$$

$$= \frac{1}{2} \left[1 + \frac{1}{2} - \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots \right]$$

$$= \frac{1}{2} \left[1 - \frac{1}{2} \right] = \frac{1}{4}$$

$$S = \log 2 - 1/2$$

3.3. Let us sum up

So far we have studied the expansion of $\log(1+x)$, $\log(1-x)$, $1 - \frac{1+x}{1-x}$,

$\log 2$ and problems solved related to these series.

3.4. Check your progress

(1) Find the sum of the series $\frac{1}{4} - \frac{1}{2} + \frac{1}{4^2} - \frac{1}{3^3} + \frac{1}{4^4} \dots \alpha$

(2) Find the sum of the series $-\frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots \alpha$

3.5. Lesson End Activities

Logarithmic series : Exercise

1) Prove that, if m and n are positive, show that

$$2 \left[\frac{m-n}{m+n} + \frac{1}{n} \left(\frac{m-n}{m+n} \right)^3 + \dots \right] = \log \left(\frac{m}{n} \right)$$

2) If α and β are the roots of the equation $x^2 - bx + c = 0$, show that

$$1 + (b + \alpha^2) \frac{1}{\alpha} + (b + \beta^2) \frac{1}{\beta} = \frac{b^2 + c}{c} + \frac{b^2 + c}{c} \frac{1}{\alpha\beta} = \frac{b^2 + c}{c} \left(1 + \frac{1}{\alpha\beta} \right)$$

3) If $n > 1$, prove that

$$\frac{1}{n+1} + \frac{1}{2(n+1)^2} + \frac{1}{3(n+1)^3} + \dots = \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \frac{1}{4n^4} + \dots$$

4) Show that

$$\frac{1}{3} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 3} + \dots = \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 2} + \dots$$

5) For positive values of x

$$2 \left[\frac{1}{2x+3} + \frac{1}{(x+1)^3} + \frac{1}{(x+1)^5} + \dots \right]$$

6) Show that $1 - \left(\frac{n+1}{n}\right)^n = \frac{1}{2} + \frac{1}{3(n+1)^3} + \frac{1}{5(n+1)^5} + \dots$

7) If $x < 1$, show that

$$2 \left[\frac{2x}{1+x^2} + \frac{1}{(1+x^2)^3} + \frac{1}{(1+x^2)^5} + \dots \right] = \frac{1}{2} \log \left[\frac{1+x}{1-x} \right]^2$$

8) Show that when $n > 1$

$$\log \frac{n+1}{n-1} = \frac{2n}{n+1} + \frac{1}{3(n+1)^3} + \frac{1}{5(n+1)^5} + \dots$$

9) If $-1 < x < 1/3$ prove that

$$2 \left[x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \right] + \frac{1}{2x} \left[\frac{x^2}{1-x} + \frac{x^4}{1-x^3} + \dots \right] = \frac{1}{1-x}$$

10) If x is positive, show that

$$\frac{x-1}{x+1} + \frac{x^2-1}{2(x+1)^2} + \frac{x^3-1}{3(x+1)^3} + \dots = \log x$$

11. Show that $1 + 1 + \left(\frac{1}{2}\right) + \frac{1}{3} + \left(\frac{1}{4}\right) + \frac{1}{4} + \left(\frac{1}{5}\right) + \frac{1}{4} + \left(\frac{1}{6}\right) + \frac{1}{7} + \left(\frac{1}{4}\right) = 1 + \sqrt{2}$

12. Show that

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \dots \alpha \cdot 10 \{$$

13. Prove the result

$$10 \log_{10} 100 = 2 \log_{10} 10$$

14. PT $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \dots \alpha \cdot 10 \log 2$

3.6. Points for Discussion

15. Sum to infinity, the series

$$\frac{5}{1 \cdot 2 \cdot 3} + \frac{7}{3 \cdot 4 \cdot 5} + \frac{9}{5 \cdot 6 \cdot 7} \dots \alpha \cdot 3 \log 2$$

16. Prove that

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{5 \cdot 6 \cdot 7} \dots \alpha \cdot 10 \log \frac{2}{2}$$

17. Prove that

$$\frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{4 \cdot 5 \cdot 6} + \frac{1}{6 \cdot 7 \cdot 8} \dots \alpha \cdot \frac{3}{4} \log$$

18. Prove that

$$\frac{7}{1 \cdot 2 \cdot 3} + \frac{1}{3 \cdot 4 \cdot 5} + \frac{9}{5 \cdot 6 \cdot 7} \dots \alpha \cdot 10 \log 2$$

3.7. References

A Text Book of Algebra by T.K. Manickavasagam Pillai

Lesson-4

CONVERGENCE AND DIVERGENCE OF SERIES

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4.0 Aims and Objectives

The Aim is to learn a new concept known as convergency and divergency of series in Algebra. Also we are going to learn about oscillating series finitely and infinitely. Comparison test is a test to know whether the given series is convergent or divergent. Also Candey's Root test, D' Alembert's ratio test also helps us in finding a series whether Convergent or divergent.

4.1 Convergence and divergence of series

Definition 1: Sequence : If to the integers $1,2,3,\dots,n$ there correspond definite number u_1,u_2,u_3,\dots,u_n then the set u_1,u_2,\dots,u_n is called a sequence and is denoted by u_n .

Definition 2: Limit of a sequence

A sequence u_n is said to tend towards a finite limit if the successive terms of the sequence approach a definite number l in a way that the difference $|u_n-l|$ becomes less than any positive quantity.

Definition 3: A sequence which tends to a finite limit is said to converge and is called a convergent sequence.

Eg: (1) Consider the sequence (u_n) where

$$u_n = \frac{n-1}{2n+3} \lim_{n \rightarrow \infty} \frac{1}{2}$$

This sequence is convergent

Note : A sequence $[u_n]$ cannot converge to two distinct limits l and l^1 .

Definition 4: A sequence which converges to zero is called a null sequence

Definition 5: Series : An expression of the form $u_1+u_2+u_3+\dots+u_n+\dots$ where for every positive integral value of n , there is a definite value for u_n is called a series.

Definition 6: If each term of the series is followed by another, it is called an infinite series.

Definition 7: If the series terminates after a finite number of terms, it is called a finite series.

Definition 8: $u_1+u_2+u_3+\dots+u_n$ is called the sum to n terms of the series. This is denoted by s_n .

Definition 9: If s_n tends to a finite limit ‘&’ as n tends to α then the series is said to be a convergent series.

Definition 10: If $\lim_{n \rightarrow \alpha} s_n = \pm \infty$ then the series is said to be a divergent series.

Definition 11: If a series does not converge to any finite limit nor diverge to plus infinite or minus infinity, then the series is said to oscillate.

4.2. Example

Convergent Series

Consider the series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \alpha$$

$$s_n = \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2 - \frac{2}{2^{n-1}}$$

As $n \rightarrow \infty$, $\frac{1}{2^{n-1}} \rightarrow 0$

$\therefore \lim_{n \rightarrow \infty} s_n = 2$

\therefore The series is convergent

Divergent

Consider the series $\sum_{n=1}^{\infty} u_n$ where $u_n = n$

$$s_n = 1 + 2 + 3 + \dots + n$$

$$= \frac{n(n+1)}{2}$$

$\lim_{n \rightarrow \infty} \frac{n(n+1)}{2} = \infty$

3. Oscillating series

(a) Oscillating finitely series

Consider the series $\sum_{n=1}^{\infty} (-1)^{n+1}$

$$S_n = 1 - 1 + 1 - 1 + 1 - 1 + \dots \text{ n terms}$$

$$s_n = 1 \text{ if } n \text{ is odd}$$

$$= 0 \text{ if } n \text{ is even}$$

$\therefore \sum (-1)^{n+1}$ oscillates finitely

(b) Oscillating infinitely series

Consider the series $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot n$

$$s_n = 1 - 2 + 3 - 4 + 5 - 6 + \dots \text{ n terms}$$

$$s_n = -1/2 n \text{ if } n \text{ is even}$$

$$s_n = 1/2(n+1) \text{ if } n \text{ is odd}$$

As $n \rightarrow \infty$, $s_n \rightarrow -\infty$ if n is even and

$$s_n \rightarrow \infty \text{ if } n \text{ is odd}$$

$\therefore \sum (-1)^{n+1} n$ oscillates infinitely

Elementary Results

The Geometric series

$1+x+x^2+\dots+x^{n-1}+\dots$ is convergent if $|x|<1$ and diverges if $x < -1$

The Geometric series

$1+x+x^2+x^3+\dots+x^n+\dots\alpha$

$$s_n = \frac{1-x^{n+1}}{1-x} \neq 1$$

If $|x| < 1$, then $x^n \rightarrow 0$, as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} s_n = \frac{1}{1-x}$$

If $n \geq 1$, $s_n \rightarrow \alpha$

If $x = -1$, the series is

$1-1+1-1+1-1+\dots$

$s_n = 1$ if n is odd

$= 0$, if n is even

$\sum_{n=0}^{\infty} x^n$ oscillates finitely when $x = -1$.

where $x < -1$, $s_n \rightarrow \infty$ or α according as n is odd or even

$\sum_{n=0}^{\infty} x^n$ oscillates infinitely

The series $\sum_{n=0}^{\infty} x^n$ converges if $|x| < 1$, diverges if $x \geq 1$, $n=0$ oscillates finitely

if $x = -1$ and oscillates infinitely if $x < -1$

(2) If $u_1+u_2+u_3+\dots+u_n+\dots$ is convergent and has the sum s , then $u_{m+1}+u_{m+2}+\dots+\dots$ is convergent and has the sum $s-(u_1+u_2+\dots+u_m)$ where $m \in \mathbb{N}$.

(3) The $u_1+u_2+\dots+u_n+\dots$ is convergent and has the sum s , then

$ku_1+ku_2+ku_3+\dots+ku_n+\dots$ converge to ks

(4) If u_n converges to s , $\sum v_n$ converges to l , then $\sum (u_n+v_n)$ converges to $s+l$

(5) A series of positive terms cannot oscillate. It is either convergent or divergent

(6) The series $\sum u_n$ is convergent, then $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

The converse is not true.

Consider the series $\sum \frac{1}{n}$

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$$

$S_n \rightarrow \infty$ as $n \rightarrow \infty \therefore \sum U_n$ is divergent

\therefore when $\lim_{x \rightarrow \infty} \frac{1}{x} = 0$ does not imply

$\sum U_n$ is convergent.

4.3 Comparison test

Form I: Let $\sum a_n$ be a series of positive terms. Let $a_n \leq k b_n$ for all n .

Then the series $\sum a_n$ is convergent if $\sum b_n$ is convergent.

(ii) Let $a_n \geq k b_n$, $\forall n$, then the series $\sum a_n$ is divergent if $\sum b_n$ is divergent.

Proof: Given $a_n \leq k b_n$

$$a_1 \leq k b_1$$

$$a_2 \leq k b_2$$

$$a_3 \leq k b_3$$

.....

.....

$$a_n \leq k b_n$$

$$\therefore a_1+a_2+\dots+a_n \leq k (b_1+b_2+\dots+b_n)$$

$$\leq k B$$

∴ $S_n \leq$ where x is a constant

∴ S_n is bounded and $\sum a_n$ is convergent if $\sum b_n$ is convergent

Step 2: Given $a_n \geq k b_n$

$$\therefore a_1 \geq k b_1$$

$$a_2 \geq k b_2$$

$$a_3 \geq k b_3$$

.....

$$a_n \geq k b_n$$

$$\therefore a_1+a_2+\dots+a_n \geq (b_1+b_2+\dots+b_n)$$

$$S_n \geq N \text{ where } n \geq m$$

$$\geq \mu \text{ where } \mu \text{ is a constant}$$

∴ $\sum a_n$ is divergent.

FIRM II If Numbers L and M exists such that $L \leq \frac{a_n}{b_n} \leq M$ for all n, then

$\sum a_n$ and $\sum b_n$ both converge or diverge together.

Proof $\frac{a_n}{b_n} \leq M$

$$a_n \leq M b_n$$

$\sum a_n$ is convergent if $\sum b_n$ is convergent.

$$L \leq \frac{a_n}{b_n}$$

$$Lb_n \leq a_n$$

$\therefore \sum a_n$ is divergent if $\sum b_n$ is divergent.

FORM III

If

$$\lim_{x \rightarrow \infty} \frac{u_n}{v_n} = k, \text{ a finite number then}$$

(i) $\sum u_n$ is convergent if $\sum v_n$ is convergent

(ii) $\sum u_n$ is divergent if $\sum v_n$ is divergent.

Proof:

$$\lim_{x \rightarrow \infty} \frac{u_n}{v_n} =$$

$$\therefore \text{for all } n \geq N, \left| \frac{u_n}{v_n} - k \right| < \epsilon$$

where ϵ is any positive number however small. $\frac{u_n}{v_n} < k + \epsilon$

$$\therefore \frac{u_n}{v_n} < k + \epsilon < R \text{ where } R_1$$

Some finite positive number $\therefore \sum u_n$ is convergent if $\sum v_n$ is convergent
we have

$$\text{ii) } \left| \frac{u_n}{v_n} - k \right| < \epsilon, \forall n \geq N$$

$$\therefore \frac{u_n}{v_n} > k - \epsilon$$

As $k > 0$, ϵ can be so chosen that $k - \epsilon > k^1 > 0$ where k^1 is a finite number.

$\therefore \sum u_n$ is divergent if $\sum v_n$ is divergent

4.4 D' Alemberts Ratio test

let a_n be a series of positive terms

i) If $\frac{a_{n+1}}{a_n} < k < 1$, for all n , $\sum a_n$ is convergent

ii) If $\frac{a_{n+1}}{a_n} \geq 1$, for all n , $\sum a_n$ is divergent.

Proof : By data $\frac{a_{n+1}}{a_n} < k < 1, \forall n$ a_{n+1}

$$\therefore \frac{a_2}{a_1} < k$$

$$\therefore a_2 < k a_1$$

$$\frac{a_3}{a_2} < k \therefore a_3 < k a_2$$

$$a_3 < k^2 a_1$$

$$a_4 < k^3 a_1$$

$$a_{n+1} < k^n a_1$$

$$\text{ie. } a_n < k^{n-1} a_1$$

$$\therefore a_n < k^{n-1} a_1$$

$$\text{But } k^{n-1} = a_1(1+k+k^2+\dots+k^{n-1})$$

This is a Geometric series with common Ratio = $k < 1$

\therefore By Comparison test, a_n is convergent.

$$\text{(ii) } \frac{a_{n+1}}{a_n} \geq 1, \forall n$$

$$\therefore a_{n+1} \geq a_n$$

$$\therefore a_2 \geq a_1$$

$$a_3 \geq a_2 \geq a_1$$

$$a_4 \geq a_3 \geq a_2 \geq a_1$$

$$\therefore a_{n+1} \geq a_1$$

$$\therefore a_2 + a_3 + a_4 + \dots + a_{n+1} \geq a_1 + a_1 + \dots (n-1) \text{ terms} = (n-1)a_1$$

$$\therefore a_1 + a_2 + a_3 + \dots + a_{n+1} \geq n a_1$$

i.e., $a_1 + a_2 + a_3 + \dots + \dots \geq N$ if $n \geq m$

$\therefore a_n$ is divergent.

Hence if $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = k$, then a_n is convergent if $k < 1$ and is

divergent if $k > 1$.

Theorem : The series $\frac{1}{n^p}$ is convergent if $p > 1$ and is divergent if $p \leq 1$

Proof : $\frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{n^p} + \dots \infty$

Case (i) $p > 1$

Choose V_n such that

$$V_1 = \frac{1}{1^p}$$

$$V_2 = V_3 = \frac{1}{2^p}$$

$$V_4 = V_5 = V_6 = V_7 = \frac{1}{4^p}$$

$$\therefore V_n = \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{2^p}\right) + \left(\frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p}\right) + \frac{1}{4^p} + \dots \infty \dots$$

$$= \frac{1}{1^p} + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \dots \infty$$

The terms of u_n are equal to or less than those of v_n

$$V_n = \frac{1}{1^p} + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 \dots \infty$$

This is a Geometric series with common ratio = $\frac{1}{2^{p-1}}$

Now $p > 1$

$$\therefore 2^{p-1} > 1$$

$$\therefore \frac{1}{2^{p-1}} < 1$$

$$\therefore \frac{1}{2^{p-1}} \text{ is convergent}$$

∴ i.e V_n is convergent

∴ By comparison test, u_n is convergent.

Case (ii) $p = 1$

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots$$

Form V_n such that

$$V_1 = 1$$

$$V_2 = \frac{1}{2}$$

$$V_3 = V_4 = \frac{1}{4}$$

$$V_5 = V_6 = V_7 = V_8 = \frac{1}{8}$$

.....

.....

$$\begin{aligned} V_n &= 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) \frac{1}{8} + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) \frac{1}{8} + \dots \infty \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \infty \end{aligned}$$

$$\therefore S_n = 1 + (n-1) \times \frac{1}{2}$$

$$S_{n+1} = 1 + \frac{n}{2} > N \text{ if } n \geq m$$

∴ By comparison test, u_n is divergent.

Case 3 If $p < 1$

$$u_n = \frac{1}{1^p} + \frac{1}{2} + \frac{1}{3} + \dots \dots \dots \infty \dots \dots \dots (1)$$

We shall assume for this series, $p = 1$

$$\therefore V_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \dots \dots (2)$$

Comparing (1) and (2)

We see that except the first term all other terms are $> V_n$

But (2) is divergent

\therefore By comparison test,

u_n is divergent.

4.5 Examples on Comparison Test

1. Test the convergence of $\sum_1^{\infty} \frac{1}{1 + 3^n}$

$$u_n = \frac{1}{1 + 3^n}$$

Choose $V_n = \frac{1}{n^4}$

$$\begin{aligned} \therefore u_n &= \frac{n^4}{1 + 3^n} = \frac{\cancel{n^4}}{n^4 \left(\frac{1}{n^4} + 3 \right)} \\ &= \frac{1}{\frac{1}{n^4} + 3} \end{aligned}$$

$$\lim_{n \rightarrow \infty} u_n = \frac{1}{3} \neq 0$$

But $V_n = \frac{1}{n^4}$ which is of the form $\frac{1}{n^p}$, $p=4 > 1$, which is convergent.

\therefore By comparison test, U_n is convergent.

$$2. \sum_1^{\infty} [\sqrt{n^4 + n^2}]^{-1} n$$

$$u_n = \sqrt{n^4 + n^2}^{-1} n$$

$$= \sqrt{n^4 + n^2}^{-1} n \times \frac{\sqrt{n^4 + n^2}^{-1} n}{\sqrt{n^4 + n^2}^{-1} n}$$

$$= \frac{n^4 + n^4 - n^2}{\sqrt{n^4 + n^2}^{-1} n}$$

$$= \frac{n}{\sqrt{n^4 + n^2}^{-1} n}$$

Choose $V_n = \frac{1}{n}$

$$\frac{u}{v} = \frac{n}{n} \frac{n}{\sqrt{n^4 + n^2}^{-1} n} \times \frac{n}{1}$$

$$= \frac{n^2}{\sqrt{n^4 + n^2}^{-1} n}$$

$$= \frac{n^2}{n^2 \left[\sqrt{1 + \frac{1}{n^3}} \right]}$$

$$= \frac{1}{\sqrt{1 + \frac{1}{n^3}}}$$

$$\lim_{n \rightarrow \infty} \frac{u}{v} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1 + \frac{1}{n^3}}} = \frac{1}{2} > 0$$

But $V_n = \frac{1}{n}$ is divergent

\therefore By comparison test, u_n is divergent

3. Test the convergence of the series.

$$\sum_1^{\infty} \frac{\sqrt{n+1} - \sqrt{1}}{n^p}$$

$$u^n = \frac{\sqrt{n+1} - \sqrt{1}}{n^p}$$

$$= \frac{\sqrt{n+1} - \sqrt{1}}{n^p} \times \frac{\sqrt{n+1} + \sqrt{1}}{\sqrt{n+1} + \sqrt{1}}$$

$$= \frac{n+1 - 1}{n^p [\sqrt{n+1} + \sqrt{1}]}$$

$$= \frac{2}{n^p [\sqrt{n+1} + \sqrt{1}]}$$

Choose $v_n = \frac{1}{n^{p+\frac{1}{2}}}$

$$\therefore \frac{u_n}{v_n} = \frac{n^{p+\frac{1}{2}}}{n^p [\sqrt{n+1} + \sqrt{1}]}$$

$$= \frac{n^{p+\frac{1}{2}}}{n^p \sqrt{n} \left[\sqrt{1+\frac{1}{n}} + \sqrt{\frac{1}{n}} \right]}$$

$$= \frac{1}{\sqrt{1+\frac{1}{n}} + \sqrt{\frac{1}{n}}}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{2} \neq 0$$

But $\sum v_n = \sum \frac{1}{n^{p+\frac{1}{2}}}$ which is convergent if $p+\frac{1}{2} > 1 \therefore p > \frac{1}{2}$

$\sum v_n = \sum \frac{1}{n^{p+\frac{1}{2}}}$ is divergent if $p+\frac{1}{2} \leq 1$ i.e., $p \leq \frac{1}{2}$

∴ By comparison test,

$$\sum u_n \text{ is convergent if } p > \frac{1}{2}$$

and is divergent if $p \leq \frac{1}{2}$

4. Test the convergence of the series

$$\sum_1^{\infty} \frac{1}{n} \sin\left(\frac{1}{n}\right)$$

$$u_n = \frac{1}{n} \sin\left(\frac{1}{n}\right)$$

Choose $v_n = \frac{1}{n^2}$

$$\frac{u_n}{v_n} = \frac{1}{n} \sin\left(\frac{1}{n}\right) n^2$$

$$= \frac{n^2}{n} \sin\left(\frac{1}{n}\right)$$

$$= n \sin\left(\frac{1}{n}\right)$$

$$= \frac{\sin\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$$

Put $\frac{1}{n} = y$

$$\frac{u_n}{v_n} = \frac{\sin y}{y}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \neq 0 \text{ and } \lim_{n \rightarrow \infty} v_n = 0 \text{ and } v_n > 0$$

$$= 1 \text{ (formula } \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \text{)}$$

But $\sum v_n = \frac{1}{n^2}$ which is convergent

\therefore By comparison test

$\sum u_n$ is convergent.

4.6 Problems on D'Alembert & Ratio test

1. Examine the convergence of the series

$$\frac{2}{3} + \frac{2 \cdot 3}{3 \cdot 5} + \frac{2 \cdot 3 \cdot 4}{3 \cdot 5 \cdot 7} + \frac{2 \cdot 3 \cdot 4 \cdot 5}{3 \cdot 5 \cdot 7 \cdot 9} + \dots$$

Solution:

Consider 2, 3, 4, 5,..... n^{th} term = $n+1$

Consider 3, 5, 7, 9,..... n^{th} term = $2n+1$

$$u_n = \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot n \dots (n+1)}{3 \cdot 5 \cdot 7 \dots (2n+1)}$$

$$u_{n+1} = \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot n \dots (n+1) \cdot (n+2)}{3 \cdot 5 \cdot 7 \dots (2n+1) \cdot (2n+3)}$$

$$\frac{u_n}{u_{n+1}} = \frac{n+2}{2n+3}$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{n+2}{2n+3}$$

$$= \lim_{n \rightarrow \infty} \frac{n \left(1 + \frac{2}{n}\right)}{n \left(2 + \frac{3}{n}\right)}$$

$$= \frac{1}{2} < 1$$

∴ By ratio test, $\sum u_n$ is convergent.

2. Examine the convergence of the series

$$\frac{n}{1 \cdot 2} + \frac{n^2}{2 \cdot 3} + \frac{n^3}{3 \cdot 4} + \dots \infty$$

$$u_n = \frac{n^n}{n(n+1)}$$

$$u_{n+1} = \frac{n^{n+1}}{(n+1)(n+2)}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{n}{n+2} \cdot n$$

$$\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{n}{n+2} \cdot n$$

$$= \lim_{n \rightarrow \infty} \frac{n}{n \left(1 + \frac{2}{n}\right)} \cdot n$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = n$$

∴ By Comparison test, $\sum u_n$ is convergent if $n < 1$ and is divergent if $n > 1$.

Step 2 :

If $n = 1$, the test fails

$$\therefore u_n = \frac{n^1}{n(n+1)}$$

Choose $V_n = \frac{1}{n^2}$

$$\therefore \frac{u_n}{v_n} = \frac{n^2}{n(n+1)}$$

$\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1 \neq 0$ But $\sum v_n = \sum \frac{1}{n^2}$ is convergent.

\therefore By comparison test, $\sum u_n$ is convergent.

4.7 Cauchy's Root test

Statement :

If $\lim_{n \rightarrow \infty} \sqrt[n]{u_n} = l$ the $\sum u_n$ is convergent if $l < 1$, and divergent if $l > 1$.

Pf: $\lim_{n \rightarrow \infty} \sqrt[n]{u_n} = l$ where ϵ is any +ve number, however small.
 $\Rightarrow \left| \sqrt[n]{u_n} - l \right| < \epsilon \quad \forall n \geq m$

i.e., $l - \epsilon < \sqrt[n]{u_n} < l + \epsilon$

Case 1

If $l < 1$, choose a number k & $l < k < 1$

Then ϵ be chosen that $l + \epsilon < k < l$

$$\therefore \sqrt[n]{u_n} < k$$

$$\Rightarrow u_n < k^n, n \geq m$$

But $\sum k^n$ is convergent if $k < 1$

$\therefore \sum u_n$ is convergent.

Case 2

If $l > 1$, ϵ be chosen so that

$$l - \epsilon > 1$$

$$\therefore u_n^{\frac{1}{n}} > 1$$

$$\therefore u_n > 1, n \geq m$$

$$\therefore \lim_{n \rightarrow \infty} u_n \neq 0$$

$\therefore \sum U$ is divergent

4.8 Examples on Cauchy's Root Test

1. Test the convergence of the series

$$\sum_1^{\infty} \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{n^{\frac{3}{2}}}}$$

$$a_n = \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{n^{\frac{3}{2}}}}$$

$$a_n^{1/n} = \left[\frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{n^{\frac{3}{2}}}} \right]^{\frac{1}{n}}$$

$$= \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{\frac{3}{2}/n}} = \frac{1}{\left(1 + \frac{1}{\sqrt{n}}\right)^{\frac{3}{2n}}} = \frac{1}{\sqrt[n]{e}} < 1$$

$\therefore \sum U$ is convergent, by Cauchy's Root Test.

{ FORMULA USED :

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

2. Test the convergence of $\sum_1^{\infty} \frac{1}{\left(1 + \frac{1}{n}\right)^{n^2}}$

Solution : $a_n = \frac{1}{\left(1 + \frac{1}{n}\right)^{n^2}}$

$$\therefore a_n^{1/n} = \left[\frac{1}{\left(1 + \frac{1}{n}\right)^{n^2}} \right]^{1/n}$$

$$= \frac{1}{\left(1 + \frac{1}{n}\right)^{n^2/n}}$$

$$= \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$\therefore \lim_{n \rightarrow \infty} a_n^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n}$$

$$= \frac{1}{e} < 1$$

\therefore By Cauchy's root test

a_n is convergent.

3) Test the convergence of $\sum \frac{1}{(10^n)^g}$

Solution $a_n = \frac{1}{(10^n)^g}$

$$\begin{aligned}
 a_n^{1/n} &= \left[\frac{1}{(10^n)^g} \right]^{1/n} \\
 &= \frac{1}{(10^{n/g})} \\
 &= \frac{1}{10^n \xi} \\
 \therefore \lim_{n \rightarrow \infty} a_n^{1/n} &= \lim_{n \rightarrow \infty} \frac{1}{10^n \xi} \\
 &= 0 < 1
 \end{aligned}$$

∴ By Cauchy's root test

a_n is convergent.

4) Test the convergence of the series $\sum \frac{x^n}{n^n}$

solution

$$\begin{aligned}
 a_n &= \frac{x^n}{n^n} \\
 \therefore a_n^{1/n} &= \left(\frac{x^n}{n^n} \right)^{1/n} = \frac{x}{n} \\
 \therefore \lim_{n \rightarrow \infty} a_n^{1/n} &= \lim_{n \rightarrow \infty} \frac{x}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} \\
 &= 0 < 1
 \end{aligned}$$

∴ By Cauchy's root test,

a_n is convergent.

4.9 Let us sum up

We have so far learnt in how to test the convergence or divergence of the series using geometric series, comparison test, D'Alembert's ratio test, Cauchy's root test.

4.10. Check your progress

(1) Test the convergence of $\sum \frac{1}{n^2 + 1}$

(2) Test the convergence of $\sum \frac{1}{(n+1)n+2}$

4.11. Lesson End Activities

1. Examine the convergence of the series

$$\sum_{n=1}^{\infty} \frac{(n+1)(n+2)(n+3)\dots+n}{n^n}$$

2. $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$ (Convergent)

3. $\frac{n!}{n^n}$ (Convergent)

4. $\frac{2^n (n!)^2}{n^n}$ (Convergent)

Problems on Comparison test

I. Examine the convergency or divergency of the following series.

a) $\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 7} + \dots$ (convergent)

b) $\frac{1}{2^2} + \frac{2}{3^2} + \frac{3}{4^2} + \dots$ (divergent)

c) $\sum \frac{n^2 + 3}{n^3 + 2}$ (divergent)

d) $\sum \sqrt{\frac{2n^2 + 3}{5n^3 + 7}}$ (divergent)

e) $\sum_1^{\infty} \frac{\sqrt{n+1}}{n+2}$ (divergent)

f) $\sum_1^{\infty} \frac{\sqrt{n+3}}{n^p}$ (Convergent if $p > \frac{1}{2}$ and divergent if $p \leq \frac{1}{2}$)

g) $\sum_3^{\infty} [n^3 + 3n] 3^{-n}$ (convergent)

4.12 Points for discussion

Problems on Ratio test

Examine the convergency and divergency of the following series.

a) $\frac{4}{5} \cdot \frac{4}{5} \cdot \frac{7}{8} \cdot \frac{4}{5} \cdot \frac{7}{8} \cdot \frac{1}{1} \cdot \frac{0}{1} \dots$ (divergent)

b) $\frac{3}{5} \cdot \frac{1}{2} \cdot \frac{3}{5} \cdot \frac{5}{6} \cdot \frac{1}{2} \cdot \frac{3}{5} \cdot \frac{5}{6} \cdot \frac{7}{2} \cdot \frac{1}{2} \dots$ (convergent)

c) $\sum_0^{\infty} \frac{n^n}{(n+1)(n+2)}$ (Convergent if $n \leq 1$ and divergent if $n > 1$)

d) $\sum \frac{5n^5 + n!}{6n^1 + 4} \cdot n^n$ (Convergent if $n = 1$ and divergent if $n > 1$)

e) $\sum \frac{(n+2)n(3-n)}{n(n-1)} \cdot n^n$ (Convergent if $n < 1$ and divergent if $n \geq 1$)

4.13 References

A Text Book of Algebra by T.K. Manickavasagam Pillai

Lesson - 5

CONDITIONALLY CONVERGENT SERIES

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5.0 Aims and Objectives

Our Aim is to learn on Absolute convergence Alternating series and conditionally convergent.

5.1. Definition

Alternating series is a series in which the terms are alternatively positive and negative.

Theorem: An alternating series $a_1 - a_2 + a_3 - a_4 + \dots$ converges if each term is numerically less than the preceding term and the n^{th} term tends to zero as $n \rightarrow \infty$.

Proof Let the series be $u_1 - u_2 + u_3 - u_4 + \dots$ where u_1, u_2, \dots are all > 0 and

$$u_1 > u_2 > u_3 > \dots > u_n > u_{n+1} > \dots$$

Let s_n denote the sum to n terms of this series

$$\therefore S_{2n} = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \dots + (u_{2n-1} - u_{2n})$$

As $u_1 > u_2 > u_3 > \dots > u_n > u_{n+1} \dots$ We have $\therefore s_{2n} > 0$

$$S_{2n+2} = S_{2n} + (u_{2n+1} - u_{2n+2}) > S_{2n}$$

$\therefore S_{2n}$ is a monotonic increasing sequence ... (1)

$$\text{Also } s_{2n} = u_1 - (u_2 - u_3) - (u_4 - u_5) \dots - (u_{2n-2} - u_{2n-1}) - u_{2n}$$

$$\text{As } u_2 > u_3, u_4 > u_5, \dots, u_{2n-2} > u_{2n-1}$$

$$\text{We have } s_{2n} < u_1 \dots (2)$$

From (1) and (2)

We get $\{s_{2n}\}$ is a monotonic increasing sequence bounded above.

\therefore As $n \rightarrow \infty$, $S_{2n} \rightarrow$ a finite limit l (say)

$$\therefore \lim_{n \rightarrow \infty} s_{2n} = l$$

$$s_{2n+1} = s_{2n} + u_{2n+1}$$

$$\therefore \lim_{n \rightarrow \infty} s_{2n+1} = \lim_{n \rightarrow \infty} s_{2n} + \lim_{n \rightarrow \infty} u_{2n+1}$$

$$= l + 0 = l$$

\therefore whether n is odd or even,

$$\therefore \lim_{n \rightarrow \infty} s_n = l$$

The series $u_1 - u_2 + u_3 - u_4 + \dots$ is convergent.

5.2 Examples Discuss the convergent of the following:

a) $1 - 1/2 + 1/3 - 1/4 + 1/5 \dots$

Solution: The series is of the form

$$u_1 - u_2 + u_3 - u_4 + \dots \text{ Positive}$$

step 1: The terms are alternatively positive and negative.

Step 2: $u_1 = 1, u_2 = 1/2, u_3 = 1/3, u_4 = 1/4 \dots$

$$u_n > u_{n+1}$$

Step 3: $\therefore \lim_{n \rightarrow \infty} u_n = 0$

∴ The given series is convergent

(b) Discuss the convergence of $\sum_1^{\infty} (-1)^n \frac{1}{n+8}$

Solution The series is

$$-1/9 + 1/10 - 1/11 + 1/12 - \dots$$

$$-u_1 + u_2 - u_3 + u_4 \dots$$

The terms are alternatively positive and negative.

$$u_1 = 1/9, u_2 = 1/10, u_3 = 1/11, u_4 = 1/12 \dots$$

$$u_n > u_{n+1}$$

$$u_n = \frac{1}{n+8}$$

$$\lim_{n \rightarrow \infty} \frac{1}{n+8} = 0$$

∴ u_n is convergent

Note : If any one of the three conditions is not true, then the alternating series is not convergent.

5.2.1 Absolute Convergence

1. Let u_n be a series of positive and negative terms. It is said to be absolutely convergent if $\sum |u_n|$ is convergent.

2. If $\sum |u_n|$ is divergent and u_n is convergent then u_n is conditionally convergent.

Example :

1. Consider the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

This series is convergent.

And the $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ is divergent.

\therefore The given series is conditionally convergent.

2. Consider the series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \dots \infty$$

This series is absolutely convergent, since $\sum |u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$ is

convergent.

Exercise

Discuss the convergence of the following:-

1) $1 - \frac{1}{5} + \frac{1}{9} - \frac{1}{13} + \dots \infty$

2) $1 - \frac{1}{2}\sqrt{2} + \frac{1}{3}\sqrt{4} - \frac{1}{4}\sqrt{6} + \dots \infty$

3) $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n+n}$

4) $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n-1}}$

(Ans: (1) to (4) - Convergent.

Theorem

An absolutely convergent series is convergent.

Proof : Let u_n be a given absolutely convergent series.

Then by definition, $\sum |u_n|$ is convergent.

Now $u_n + |u_n| = 2u_n$ if u_n is +ve

$= 0$, if u_n is -ve.

\therefore every term of the series $\sum (u_n + |u_n|) \geq 0$ and is less than or equal to the

corresponding term of the convergence series $\sum 2|u_n|$

$\therefore \sum [u_n + |u_n|]$ is convergent.

But we know that $\sum |u_n|$ is convergent

$\therefore u_n$ is also convergent.

Note: (1) When we say that u_n is absolutely convergent, we assert that the

convergence of another series, (viz) $\sum |u_n|$

(2) The series $\sum |u_n|$ is absolutely convergent if after a certain stage

$\left| \frac{u_{n+1}}{u_n} \right| < R$, where $k \in \mathbb{N}$ may, vao day coi co con nho nay ngon lam

<http://www.freewebtown.com/nhatquanglan/index.html>

is a fixed number < 1 . (Ratio test)

(3) The series u_n is absolutely convergent if after a certain stage $\left| u_n^{1/n} \right| < k$,

where k is a fixed number < 1 (Root test)

(4) If two series are absolutely convergent they can be multiplied by term and the resulting series is also absolutely convergent.

5.3 Examples

Test for convergency the series

$$n - \frac{n^2}{\sqrt{2}} + \frac{n^3}{\sqrt{3}} - \frac{n^4}{\sqrt{4}} + \dots + \frac{n^n}{\sqrt{n}} \dots$$

Solution:

$$u_n = (-1)^{n-1} \frac{n^n}{\sqrt{n}}$$

$$u_{n+1} = (-1)^n \frac{n^{n+1}}{\sqrt{n+1}}$$

$$\frac{u_{n+1}}{u_n} = \left(-1 \right) \frac{n^{n+1}}{\sqrt{n+1}} \cdot \frac{n\sqrt{n}}{(-1)^{n-1} n^n}$$

$$= - \frac{n \cdot \sqrt{n}}{\sqrt{n+1}}$$

$$= - \frac{n \cdot \sqrt{n}}{\sqrt{n} \sqrt{1 + \frac{1}{n}}}$$

$$= \frac{-n}{\sqrt{1 + \frac{1}{n}}}$$

$$\therefore \left| \frac{u_{n+1}}{u_n} \right| = \left| \frac{-n}{\sqrt{1 + \frac{1}{n}}} \right|$$

$$= \left| \frac{|n|}{\sqrt{1 + \frac{1}{n}}} \right|$$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = 1$$

By ratio test u_n is absolutely convergent if $|n| < 1$.

$\therefore u_n$ is convergent if $|n| < 1$.

5.4 Let us sum up

We have studied the definitions of alternating series and absolute convergence of a series. With this we learn how to determine the nature of these series.

5.5 Check your progress

- (1) If a_n converges absolutely, then what can you say about a_n
 (2) Check whether the series $1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4}$ is conditionally convergent

5.6 Lesson and activities

Discuss the convergence of the following series:-

$$1) 1 - \frac{n}{1+a} + \frac{n^2}{1+a^2} - \frac{n^3}{1+a^3} + \dots + \frac{n^n}{1+a^n} - \dots \infty$$

(Convergent if $|n| < 1$)

$$2) n + \frac{1}{2} - \frac{n^2}{3} + \frac{n^3}{2} - \frac{n^4}{4} + \frac{n^5}{5} - \dots \infty$$

(absolutely convergent if $|n| < 1$)

5.7 Points for discussion

$$3) \frac{1}{1+n} - \frac{n}{1+n^2} + \frac{n^2}{1+n^3} - \dots + \frac{n^{n-1}}{1+n^n} - \dots \infty$$

(Ans: Convergent)

5.8. References

A Text Book of Algebra by T.K. Manickavasagam Pillai

UNIT II

LESSON 6

RAABE'S TEST

Contents

- 6.0. Aims and Objectives
- 6.1. Raabe's test
- 6.2. Cauchy's Condensation Test
- 6.3. Let us sum up
- 6.4. Check your progress
- 6.5. Lesson and activities
- 6.6. Points for discussion
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6.0 Aims and Objectives

We study here Raabe's test and Cauchy's condensation test in order to test the convergence.

6.1 Raabe's test

Let u_n be a series of positive terms. Let
$$l = \lim_{n \rightarrow \infty} n \left[\frac{u_n}{u_{n+1}} - 1 \right]$$

Then u_n is convergent if $l > 1$ and is divergent if $l < 1$.

Proof: Consider the series

$$\begin{aligned} \sum v_n &= \sum \frac{1}{n^p} \\ \lim_{n \rightarrow \infty} n \left[\frac{u_n}{u_{n+1}} - 1 \right] &= \lim_{n \rightarrow \infty} n \left[\frac{(n+1)^{-p}}{n^{-p}} - 1 \right] \\ &= \lim_{n \rightarrow \infty} n \left[\frac{n^p (1 + \frac{1}{n})^{-p}}{n^p} - 1 \right] \end{aligned}$$

$$= \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n} \right)^n \right]^{p-1}$$

$$= \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n} \right)^n}{\left(1 + \frac{1}{n} \right)^1}^{p-1}$$

$$= p \cdot 1^{p-1} = p$$

$$\therefore \left| n \left(\frac{u_n}{u_{n+1}} \right)^p - p \right| < \epsilon \quad \forall n \geq m_1$$

Where ϵ is any positive number, however small.

$$\text{i.e., } p - \epsilon < n \left(\frac{u_n}{u_{n+1}} \right)^p < p + \epsilon$$

similarly

$$l - \epsilon < n \left(\frac{u_n}{u_{n+1}} \right)^l < l + \epsilon \quad \forall n \geq m_2$$

If $l > 1$, choose p such that $l > p > 1$

As l can be chosen that $l - \epsilon > p + \epsilon$, we can find a number m & ϵ for all

$n \geq m$

$$n \left(\frac{u_n}{u_{n+1}} \right)^l > n \left(\frac{u_n}{u_{n+1}} \right)^p$$

$$\text{i.e., } \frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}, \frac{u_n}{u_{n+1}} < \frac{v_{n+1}}{v_n}$$

But as $p > 1$, $\sum v_n = \sum \frac{1}{n^p}$ is convergent.

$\therefore u_n$ is convergent.

If $\epsilon < 1$, choose p & l $l < p < 1$

As t can be so chosen that $l + \epsilon < p - t$ we can find a number m & ϵ form $n \geq m$,

$$n \left(\frac{u_n}{u_{n+1}} \right)^k < \frac{1}{k+1}$$

$$\frac{u_n}{u_{n+1}} < \frac{1}{k+1}$$

$$\text{i.e., } \frac{u_{n+1}}{u_n} > \frac{k+1}{k}$$

But as $p < 1$, $\sum v_n = \sum \frac{1}{n^p}$ is divergent.

$\therefore u_n$ is divergent.

6.2 Cauchy's Condensation Test

If $f(n) > 0$ and $f(n+1) \leq f(n)$ for all positive integral values of n and a is a positive integer > 1 , the series $\sum f(n)$ and $\sum a^n f(a^n)$ are both convergent or both divergent.

Proof: The terms of the series $\sum f(n)$ are grouped as follows

$$\sum f(n) = [f(1) + f(2) + \dots + f(n)] + [f(a) + f(2a) + \dots + f(a^2)] + \dots$$

$$+ [f(a^{2^1}) + f(2^{2^2}) + \dots + f(a^{2^{n-1}})] + \dots$$

The number of terms in the groups are $a, a^2 - a, a^3 - a^2, \dots, a^n - a^{n-1}, \dots$

Then $\sum f(n) = u_1 + u_2 + u_3 + \dots + u_n + \dots$

$$= u_n$$

Where $u_n = f(a^{n-1}) + f(a^{2^{n-1}}) + \dots + f(a^{2^{n-1}})$

Given $f(n+1) \leq f(n)$

$$\therefore f(n) \geq f(n+1)$$

$$\therefore f(n) \geq a \iff (f^{-1})^n \geq (f^{-1})^{n+1} \iff f \geq a$$

$$\therefore (a^n - a^{-1n}) f \geq a^n \iff (f^{-1})^n \geq a^n$$

$$a^{n-1} (a-1) f^n \geq a \iff \left(\frac{f}{a}\right)^n \geq a$$

Taking $u_n = a^{n-1} (a-1) f^n$

If $\sum a^n f^n$ is convergent, then u_n is convergent.

$$\text{Taking: } u_n \geq \left(1 - \frac{1}{a}\right) a^n f^n$$

If $\sum a^n f^n$ is divergent, then u_n is divergent.

$\therefore \sum f(n) \sum d(a, f)$ are either both convergent or both divergent.

6.3 Let us sum up

We have so far studied the theories in Raabe's test and condensation test.

6.4 Check your progress

1) When will you say Raabe's test needs to test the convergence of e series.

2) Write down Cauchy's Condensation test.

6.5 Lesson and activities

1) Examine the convergence of the series

$$\frac{1^2}{2^2} + \frac{1^2}{2^2} \cdot \frac{3}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{25}{6} \cdot \infty$$

6.6 Points for discussion

1) Examine the convergence of

$$\sum_1^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot n \cdot (2)^n}{2 \cdot 4 \cdot 6 \cdot n \cdot 2^n}$$

2) Discuss the convergence of

$$\sum_1^{\infty} n^r \cdot h^n$$

6.7 References

A Text Book of Algebra by T.K. Manickavasagam Pillai

Theory of Equations

Lesson – 7

THEORY OF EQUATIONS

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- 7.1. Relation between
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7.0 Aims and Objectives

We shall study solving equations of 3rd degree, 4th

7.1 Relation between

Roots and coefficients of an equation

Consider the equation of nth degree in x as

$$a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + a_3 x^{n-3} + \dots + a_n = 0$$

There are n roots (viz) $\alpha_1, \alpha_2, \dots, \alpha_n$

$$S_1 = \alpha_1 + \alpha_2 + \alpha_3 + \dots + \alpha_n = -\frac{a_1}{a_0}$$

$$S_2 = \sum \alpha_1 \alpha_2 = \frac{a_2}{a_0}$$

$$S_3 = \sum \alpha_1 \alpha_2 \alpha_3 = \frac{a_3}{a_0}$$

.....

$$S_n = \alpha_1 \alpha_2 \alpha_3 \dots \alpha_n = (-1)^n \frac{a_n}{a_0}$$

7.2 Examples

Model 1

1) Solve the equation

$$n^3 - 12n^2 + 39n - 28 = 0 \text{ whose roots are in Arithmetic Progression.}$$

Solution: Step 1

$$\text{The given equation is } n^3 - 12n^2 + 39n - 28 = 0 \therefore a_0 = 1, a_1 = -12;$$

$$a_2 = 39, a_3 = -28$$

Since the roots are in AP, let the roots be $a-d, a, a+d$

$$a - d = \alpha_1$$

$$a = \alpha_2$$

$$a + d = \alpha_3$$

$$S_1 = \alpha_1 + \alpha_2 + \alpha_3 = \frac{-a_1}{a_0}$$

$$a - d + a + a + d = 2$$

$$3a = 12$$

$$a = 4$$

Step 2

$$S_3 = \alpha_1 \alpha_2 \alpha_3 = \frac{-a_3}{a_0}$$

$$(a-d)a(a+d) = -28$$

$$4(16-d^2) = -28$$

$$16-d^2 = -7$$

$$d^2 = 9$$

$$d = \pm 3 \therefore d = 3, d = -3$$

\therefore When $a=4, d=3$, the roots are 1, 4, 7

When $a=4$, $d=-3$, the roots are 7, 4, 1

Model 2

Solve the equation

$$3n^3 - 26n^2 + 52n - 24 = 0, \text{ given that the roots are in GP}$$

Solution: Step 1

The given equation is

$$3n^3 - 26n^2 + 52n - 24 = 0$$

This is of the form

$$a_0n^3 + a_1n^2 + a_2n + a_3 = 0$$

$$\therefore a_0=3; a_1=-26; a_2=52; a_3=-24$$

Since the roots are in GP, let the roots be a/r , a , ar

$$\alpha_1 = a/r$$

$$\alpha_2 = a$$

$$\alpha_3 = ar$$

$$S_3 = \alpha_1 \alpha_2 \alpha_3 = \frac{-a_3}{a_0}$$

$$\frac{a}{r} \cdot a \cdot ar = \frac{2}{3}$$

$$a^3 = 8$$

$$\therefore a = 2$$

Step 2

$$S_1 = a/r + a + ar = \frac{-a_1}{a_0}$$

$$\frac{2}{r} + 2 + 2r = \frac{6}{3}$$

$$2\left(\frac{1}{r} + 1\right) = \frac{6}{3}$$

$$\frac{1+r+1}{r} = \frac{3}{3}$$

$$3+3r+3r^2 = 13r$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$3r^2 - 10r + 3 = 0$$

$$a=3; b=-10; c=3$$

$$r = \frac{10 \pm \sqrt{100 - 36}}{6}$$

$$r = 3, \frac{1}{3}$$

Step 3

When $a=2$, $r=3$, the roots are $\frac{2}{3}, 2$,

When $a=2$, $r=\frac{1}{3}$, the roots are $6, 2, \frac{2}{3}$

3. Model 3 HP

Solve the equation

$$6n^3 - 11n^2 + 6n - 1 = 0 \text{ whose roots are in HP}$$

Proof : Step 1

The equation is

$$6n^3 - 11n^2 + 6n - 1 = 0$$

This is of the form

$$a_0n^3 + a_1n^2 + a_2n + a_3 = 0$$

$$a_0 = 6; a_1 = -11; a_2 = 6; a_3 = -1$$

let the roots be $\alpha, \beta r$

Since the roots are in HP

$$\beta = \frac{2\alpha r}{\alpha + r}$$

$$\alpha + \beta + r = 2\alpha \dots\dots(1)$$

$$s_1 = \alpha + \beta + r = \frac{-a_1}{a_0} = \frac{1}{6} \dots\dots(2)$$

$$s_2 = \alpha + \beta + r = \frac{a_2}{a_0} = 1$$

$$2\alpha r + \alpha r = 1, \text{ using (1)}$$

$$3\alpha r = 1$$

$$\alpha r = \frac{1}{3} \dots\dots(3)$$

$$s_3 = \alpha + \beta + r = \frac{-a_3}{a_0} = \frac{1}{6}$$

But $\alpha r = \frac{1}{3}$

$$\therefore \frac{1}{3} \cdot \beta = \frac{1}{6}$$

$$\beta = \frac{1}{2}$$

Step 2

Put $\beta = \frac{1}{2}$ in $\dots\dots(2)$

$$\alpha + r + \frac{1}{2} = \frac{1}{6}$$

$$\alpha + r = \frac{1}{6} - \frac{1}{2} + \frac{1}{6} - \frac{8}{3} \dots\dots\dots(4)$$

Solve (3) and (4) for α, r

From (4) $r = \frac{4}{3} - \alpha$

Using in (3)

$$\alpha \left(\frac{4}{3} - \alpha \right) = \frac{1}{3}$$

$$\frac{4\alpha}{3} - \alpha^2 = \frac{1}{3}$$

$$\therefore 4\alpha - 3\alpha^2 = 1$$

$$3\alpha^2 - 4\alpha + 1 = 0$$

$$3\alpha^2 - 3\alpha - \alpha + 1 = 0$$

$$3\alpha(\alpha - 1) - (\alpha - 1) = 0$$

$$(\alpha - 1)(3\alpha - 1) = 0$$

$$\alpha = 1, \alpha = \frac{1}{3}$$

Step 3

$$\alpha = 1 \Rightarrow r = \frac{4}{3} - \frac{4}{3} = \frac{1}{3}$$

$$\therefore \alpha = \frac{1}{3}, \beta = \frac{1}{2}, r = \frac{1}{3}$$

Step 4

$$\text{When } \alpha = \frac{1}{3}, \beta = \frac{4}{3} - \frac{4}{3} - \frac{1}{3} =$$

$$\therefore \alpha = \frac{1}{3}, \beta = \frac{1}{2}, r = 1$$

∴ The roots are $\frac{1}{3}, -\frac{1}{2}$

4. Model 4

Solve $n^4 - 12n^3 + 14n^2 + 132n - 135 = 0$ given that the roots are in AP

Solution: $a_0 = 1, a_1 = -12, a_2 = 14, a_3 = 132, a_4 = -135$

Let the roots be $a - 3d, a - d, a + d, a + 3d$

$$S_1 = a - 3d + a - d + a + d + a + 3d = \frac{-a_1}{a_0}$$

$$4a = 12$$

$$a = 3$$

$$s_4 = (a - 3d)(a - d)(a + d)(a + 3d) = \frac{a_4}{a_0}$$

$$(a^2 - 9d^2)(a^2 - d^2) = -135$$

$$(9 - 9d^2)(9 - d^2) = -135$$

$$9(1 - d^2)(9 - d^2) = -135$$

$$(1 - d^2)(9 - d^2) = -15$$

Put $-d^2 = y$

$$\therefore (1 - y)(9 - y) = -15$$

$$y^2 - 10y + 24 = 0$$

$$y = 6, y = 4$$

$$\therefore d^2 = 6, d^2 = 4$$

$$d = \sqrt{6}, d = \pm 2$$

Step 2 : $a = 3, d = 2$

The four roots are

-3, 1, 5, 9

Step 3 : When a=3, d=-2

The four roots are

9, 5, 1, -3

Model 5

5. Solve

$$4n^4 - 85n^3 + 357n^2 - 340n + 64 = 0$$

whose roots are in GP

Proof: Let the roots be

$$\frac{a}{r^3}, \frac{a}{r}, a, r, a^3$$

$$a_0 = 4, a_1 = -85, a_2 = 357, a_3 = -340, a_4 = 64$$

$$S_4 = \frac{a}{r^3} + \frac{a}{r} + a + r + a^3 = \frac{a_4}{a_0}$$

$$a_4 = 64/4 = 16$$

$$a = 2$$

Step 2

$$S_1 = \frac{a}{r^3} + \frac{a}{r} + a + r + a^3 = 8 \sqrt[5]{4}$$

$$a \left(\frac{1}{r^3} + \frac{1}{r} + 1 + r + r^3 \right) = 8 \sqrt[5]{4}$$

$$2 \left[\left(r + \frac{1}{r} \right) + \left(\frac{1}{r^3} + r^3 \right) \right] = 8 \sqrt[5]{4}$$

$$\left(r + \frac{1}{r}\right) + r\left(\frac{1}{r}\right)^3 - 3\left(r + \frac{1}{r}\right) = \frac{8\sqrt{5}}{8}$$

$$\{\because a^3 + b^3 = (a+b)(a^2 - ab + b^2)\}$$

$$\left(r^3 + \frac{1}{r^3}\right) - r^2\left(\frac{1}{r}\right) - \frac{8\sqrt{5}}{8} = 0$$

Put $r + \frac{1}{r} = a$

$$a^3 - 2a - \frac{8\sqrt{5}}{8} = 0$$

$$8a^3 - 16a - 85 = 0$$

$a = \frac{5}{2}$ is a root

$$\therefore r + \frac{1}{r} = \frac{5}{2} \Rightarrow \frac{1}{r},$$

When $r=2$, $a=2$, the roots are $\frac{1}{4}, 1, 4, 16$

When $r = \frac{1}{2}$, $a=2$, the roots are $16, 4, 1, \frac{1}{4}$,

Model 6

6. Solve : $n^4 - 2n^3 + 4n^2 + 6n - 21 = 0$ given that it has two roots equal in magnitude but opposite in sign.

Proof : $a_0=1, a_1=-2, a_2=4, a_3=6, a_4=-21$

Let the roots be $\alpha, \beta, \gamma, \delta$

Given $\alpha + \beta = 0$; $\therefore \alpha = -\beta$

$$S_1 = \alpha + \beta r + \delta = \frac{-a_1}{a_0}$$

$$r + \delta = 2 \dots\dots\dots(1)$$

$$S_2 = \alpha + \beta(\alpha + \alpha + \delta) + \beta r + \delta = \frac{a_2}{a_0}$$

$$\alpha + \beta(2\alpha + \delta) + \beta r + \delta = 4$$

Put $r + \delta = 2$

$$\therefore \alpha + \beta(2\alpha + \delta) + \beta r + \delta = 4$$

$$\alpha + \beta r + \delta = 4 \dots\dots\dots(2) \quad \{\because \alpha + \beta = 0\}$$

$$S_3 = \alpha + \beta(\alpha + \beta + \delta) + \beta r + \delta = \frac{-a_3}{a_0}$$

$$\alpha + (\beta + \delta) + \delta = -6$$

Put $r + \delta = 2$, $\alpha + \beta = 0$

$$\therefore 2\alpha = \beta - 6$$

$$\alpha + \beta = 3 \dots\dots\dots(3)$$

But $\alpha + \beta = 0$

$$\therefore \beta = -\alpha$$

$$\alpha(-\alpha^2) = -3$$

using in (3) : $\alpha^2 = 3$

$$\alpha = \pm\sqrt{3}$$

$$\alpha = \sqrt{3}, -\sqrt{3}$$

Step 2 : $r + \delta = 2$

$$S_4 = \alpha + \beta = \frac{a_4}{a_0}$$

$$\begin{aligned} -3r\delta &= 2 \\ r\delta &= 7 \end{aligned}$$

But $\delta = 2 - r$

$$\therefore r(2 - r) = 7$$

$$2r - r^2 = 7$$

$$r^2 - 2r + 7 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 28}}{2}$$

$$= \frac{2 \pm \sqrt{-24}}{2}$$

$$= \frac{2 \pm \sqrt{24}i}{2}$$

$$= \frac{2 \pm \sqrt{4 \cdot 6}i}{2}$$

$$= \frac{2 \pm 2\sqrt{6}i}{2}$$

$$r = 1 \pm i\sqrt{6}$$

$$\therefore r = 1 \pm i\sqrt{6}$$

Hence the roots are

$$1 \pm i\sqrt{6}$$

7.3 Let us sum up

We have studied so far how to solve equations whose roots are in AP, GP, HP or a relation between any of its two roots is given.

7.4 Check your progress

(1) If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4+px^3+qx^2+rx+s=0$, find the value of

$$\sum \frac{1}{\alpha}$$

(2) If α, β, γ are in HP then find β ?

(3) If α, β, γ are the roots of the equation $x^3+px^2+qx+r=0$ find $\sum \alpha^2 \beta$

7.5 Lesson end activities

1. If the roots of the equation

$$n^3+pn^2+qn+r=0 \text{ are in AP, show that } 2p^3+27r=9pq$$

2. Solve the equation

$$n^3-12n^2+39n-28=0 \text{ whose roots are in AP (Ans. 1, 4, 7)}$$

3. Solve the equation

$$n^4+2n^3-25n^2-26n+120=0 \text{ given that the product of two of its roots is 8}$$

$$\text{(Ans. 4, 2, -5, -3)}$$

4. Solve the equation

$$2n^3-n^2-22n-24=0 \text{ two of whose roots are in the ratio 3 : 4}$$

$$\text{(Ans. -3/2, -2, 4)}$$

5. If the roots of the equation

$$an^3+bn^2+cn+d=0 \text{ are in GP, show that } c^3a = b^3d$$

6. Solve the equation

$$27n^3+42n^2-28n-8=0 \text{ whose roots are in GP (Ans. -2/9, 2/3, -2)}$$

7.6 Points for discussion

1. Solve the equation

$$6n^3 - 11n^2 - 3n + 2 = 0 \text{ given that the roots are in Harmonic Progression}$$

(Ap.2003, BU)

2. Solve the equation

$$n^5 - n^4 + 8n^2 - 9n - 15 = 0 \text{ if } \sqrt{3} \text{ and } 1 - 2i \text{ are two of its roots}$$

(Ap. 2003, BU)

3. Solve $8n^4 - 90n^3 + 315n^2 - 405n + 162 = 0$ given that the roots are in Geometric Progression (Ap. 2003, BU)

4. Solve the equation $n^4 - 2n^3 + 4n^2 + 6n - 21 = 0$ given that two of its roots are equal in magnitude but opposite in sign. (Ap. 2003, BU)

5. Solve $n^4 - 2n^3 + 4n^2 + 6n - 21 = 0$ if the sum of two roots is zero. (Ap.2004,)

6. Solve the equation $4n^3 - 24n^2 + 23n + 18 = 0$ given that the roots are in Arithmetic Progression (Nov. 2003, BU)

7. Solve the equation $n^3 - 4n^2 - 7n + 10 = 0$ whose roots α and β are connected by the relation $\beta = 2\alpha + 5$ (Nov. 2003, BU)

8. The root of the equation $n^6 - 4n^5 - 11n^4 + 40n^3 + 11n^2 - 4n - 1 = 0$ is $\sqrt{2} - \sqrt{3}$
Find the remaining roots (Ap. 2005)

9. Find the value of R for which the equation $2n^3 + 6n^2 + 5n + R = 0$ are in Arithmetic Progression (Nov. 2005, BU)

7.7 References

A Text Book of Algebra by T.K. Manickavasagam Pillai

TRANSFORMATION OF EQUATIONS

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8.0 Aims and Objectives

Our aim is to learn about getting the equations whose roots are decreased or increased by a given quantity. Further we also study on how to remove the second term from the given equations.

8.1. Transformation of Equations

8.2 Examples Model 1

1. Find the equation whose roots are the roots of $n^4 - 5n^3 + 7n^2 - 17n + 11 = 0$ each diminished by 2

Solution : Let the roots be

$$\alpha, \beta, \gamma, \delta$$

We shall form the equation whose roots are $\alpha - 2, \beta - 2, \gamma - 2, \delta - 2$

By synthetic division

n=2	1	-5	7	-17	11	
		2	-6	2	-30	
	1	-3	1	-15	-19 = A ₄	
n=2		2	-2	-2		
	1	-1	-1	-17 = A ₃		
n=2		2	2			
n=2	1	1	1 = A ₂			
		2				
	1	3 = A ₁				
	A ₀					

∴ The required equation is

$$A_0n^4 + A_1n^3 + A_2n^2 + A_3n + A_4 = 0$$

$$n^4 + 3n^3 + n^2 - 17n - 19 = 0$$

Model 2

2. Find the equation whose roots are the roots of $4n^5 - 2n^2 + 7n - 3 = 0$, each increased by 2

Solution : Let the roots be $\alpha, \beta, \gamma, \delta, t$.

To form the equation whose roots are $\alpha + 2, \beta + 2, \gamma + 2, \delta + 2, t + 2$,

Increase the roots by 2 means that diminish the roots by -2

By synthetic division

n=-2	4	0	0	-2	7	-3	
	0	-8	16	-32	68	-150	
n=-2	4	-8	16	-34	75	-153	A_5
	0	-8	32	-96	260		
	4	-16	48	-130	335	A_4	
n=-2	0	-8	48	-192			
	4	-24	96	-322	A_3		
n=-2	0	-8	64				
	4	-32	160	A_2			
n=-2	0	-8					
	4	-40					
	A_0	A_1					

The transformed equation is

$$A_0n^5 + A_1n^4 + A_2n^3 + A_3n^2 + A_4n + A_5 = 0$$

$$4n^5 - 40n^4 + 160n^3 - 322n^2 + 335n - 153 = 0$$

Model 3

3. If α, β, γ are the roots of $n^3 + 2n^2 - 3n - 5 = 0$, form the equation whose roots

are $\frac{\alpha - 1}{\alpha + 2}, \frac{\beta - 1}{\beta + 2}, \frac{\gamma - 1}{\gamma + 2}$

Solution :

Let $y = \frac{n-1}{n+1}$ Imp: replace α by n

$y(n+1) = n-1$

$yn + y = n - 1$

$y + 1 = n - yn$

$y + 1 = n(1 - y)$

$\frac{y + 1}{1 - y} = n$

$\therefore n = \frac{1 + y}{1 - y}$

using this in the given equation

$\left(\frac{1+y}{1-y}\right)^3 + 2\left(\frac{1+y}{1-y}\right)^2 - 3\left(\frac{1+y}{1-y}\right) - 5 = 0$

On Simplification

$y^3 - 6y^2 - 27y + 5 = 0$

4. If α, β, γ be the roots of $n^3 - pn^2 + qn - r = 0$ form the equation whose roots are

$\beta r + \frac{1}{\alpha}, \alpha r + \frac{1}{\beta}, \beta r + \frac{1}{\gamma}$

Solution:

$\beta r + \frac{1}{\alpha} = \frac{\alpha \beta r + 1}{\alpha} \dots\dots\dots(1)$

The given equation is

$$n^3 - pn^2 + qn + r = 0 \dots\dots\dots(1)$$

Let the roots be α, β, r

$$\alpha + \beta + r = p,$$

$$\alpha\beta + r\alpha + \beta = q$$

$$\alpha\beta = \dots\dots\dots(2)$$

using (2) in (1)

$$\beta = \frac{q - r\alpha}{\alpha}$$

$$r\alpha + \frac{q - r\alpha}{\beta} = \frac{q - r\alpha}{\beta}$$

$$\alpha = \frac{q - r\alpha}{r}$$

$$\therefore y = \frac{r+1}{n}$$

$$n = \frac{r+1}{y}$$

using in (1)

$$\left(\frac{r+1}{y}\right)^3 - p\left(\frac{r+1}{y}\right)^2 + q\left(\frac{r+1}{y}\right) - r = 0$$

On simplification

$$ry^3 - q(r+1)y^2 + p(r+1)^2y - (r+1)^2 = 0$$

5. If α, β be the roots of $n^3 - pn^2 + qn - r = 0$ form the equation whose roots are

$$\frac{\alpha}{\beta + r\alpha} = \frac{\beta}{\alpha + \beta} = \frac{r}{\alpha + \beta - r}$$

Solution : Let α, β, r be the roots of

$$n^3 - pn^2 + qn - r = 0 \dots\dots\dots(1)$$

$$\alpha + \beta + r = p \dots\dots\dots(2)$$

$$\alpha + \beta + r = p \dots\dots\dots(1)$$

$$\alpha + \beta = p - r \dots\dots\dots(3)$$

$$\frac{\alpha}{\beta + r\alpha} = \frac{\alpha}{\alpha + r + \beta - \alpha}$$

$$= \frac{\alpha}{p - 2\alpha}$$

$$\frac{\beta}{r + \alpha} = \frac{\beta}{r + \alpha + \beta - \beta} = \frac{\beta}{p - \beta}$$

$$\frac{r}{2 + \beta} = \frac{r}{2 + \beta + r - r} = \frac{r}{p - r}$$

$$\therefore y = \frac{n}{p - 2n}$$

$$y(p - 2n) = n$$

$$yp - 2yn = n$$

$$yp = 2yn + n$$

$$= n(2y + 1)$$

$$n = \frac{y p}{2y + 1}$$

Using in (1)

$$\left(\frac{y}{2y+1}\right)^3 p^3 - p \left(\frac{y^2 p}{2y+1}\right) + q \left(\frac{y p}{2y+1}\right) - r = 0$$

On simplification

$$y^3(4pq-8r-p^3)+y^2(4pq-12r-p^3)+y(pq-br)-r=0$$

6. If α, β, r be the roots of $n^3+qn+r=0$, form the equation whose roots are

$$\beta + r, \alpha - 2r, \beta - 2r, \alpha + r, \beta - 2r$$

Solution: The given equation is

$$n^3+qn+r=0 \dots\dots\dots(1)$$

$$\therefore a_0=1, a_1=0, a_2=q, a_3=r$$

$$\alpha + \beta + r = -\frac{a_2}{a_0} = 0$$

$$\beta + r = -\alpha \quad \beta - 2r = -\alpha \quad 3 - \alpha = 3\alpha$$

$$\text{III}^{\text{ly}} \quad r + \alpha - 2\beta = 3\beta$$

$$2 + \beta - 2r = 3r$$

$$\therefore y = -3n$$

$$\therefore n = \frac{-y}{3}$$

$$\text{Using in (1), } \left(\frac{-y}{3}\right)^3 + q\left(\frac{-y}{3}\right) + r = 0$$

$$y^3 + 9qy - 27r = 0$$

Model 4: Removal of second term

1. Consider the n^{th} degree equation in n as

$$a_0x^n + a_1x^{n+1} + a_2x^{n-2} + \dots + a_{n-1}x^{n-1} + a_n = 0$$

We shall remove the second term by diminishing (decreasing) the roots of

this equation by $h = -\frac{a_1}{n_0 a}$

Example : Remove the second form from $n^5 + 5n^4 + 3n^3 + n^2 + n + 1 = 0$

Pf.: $a_0=1, n=5, a_1=5$

$$h = \frac{-a_1}{n_0 a} = \frac{5}{5} = 1$$

$\therefore n=-1$	1	5	3	1	1	1	
	0	-1	-4	1	-2	1	
	1	4	-1	2	-1	2	$=A_5$
$n=-1$	0	-1	-3	4	-6		
	1	3	-4	6		-7	$=A_4$
$n=-1$		-1	-2	6			
	1	2	-6		12		$=A_3$
$n=-1$		-1	-1				
	1	1		-7			$=A_2$
$n=-1$		-1					
	1	0					

$A_0 \quad A_1$

The transformed equation is

$$n^5 - 7n^3 + 12n^2 - 7n + 2 = 0$$

We notice that the second term (viz.,) the term containing n^4 is removed.

8.3. Let us sum up

We have studied so far how to get an equation whose roots are decreased or increased by a given quantity.

8.4 Check your progress

1. If α, β, γ be the roots of $ax^3+bx^2+cx+d=0$, then what is the value of

$$\sum \alpha^2$$

2. Diminish the roots of $x^4-5x^3+7x^2-17x+11=0$ each diminished by 2

$$(\text{Ans } x^4+3x^3+x^2-17x-19=0)$$

8.5 Lesson end activities

1. If α, β, γ are the roots of the equation $n^3+an^2+bn+c=0$, form the equation whose roots are $\alpha + \beta, \beta + \gamma$ and $\gamma + \alpha$ (Nov. 2006, BU)

2. If α, β, γ are the roots of $n^3+pn^2+qn+r=0$ form the equation whose roots are $\beta + r, \gamma + r, \alpha - r$ and $\alpha + \beta - 2r$ (Ap. 1999)

3. If α, β, γ are the roots of $n^3+pn^2+qn+r=0$ form the equation whose roots are $\alpha - \frac{1}{\beta}, \beta - \frac{1}{\alpha}, \gamma - \frac{1}{\alpha}$ (Ap.1997)

4. Diminish the roots by 1 of the equation $n^4-3n^3+4n^2-2n+1=0$ and hence solve the same. (Nov.1997, BU)

8.6 Points for discussion

5. If α, β, γ are the roots of $n^3+pn^2+qn+r=0$ form the equation whose roots are $\alpha(\beta + \gamma), \beta(\gamma + \alpha), \gamma(\alpha + \beta)$ (Nov.2005)

6. Diminish by 2 the roots of the equation $n^4-5n^3+7n^2-17n+11=0$
(Ans. $y^4+3y^3+y^2-17y-19=0$)

8.7 References

A Text Book of Algebra by T.K. Manickavasagam Pillai.

Lesson-9

RECIPROCAL EQUATIONS

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- 9.0. Aims and Objectives
- 9.1. Reciprocal Equations
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9.0 Aims and Objectives

We shall study the completely a reciprocal equation and how to solve the same and study the nature of roots of a given equation.

9.1 Reciprocal Equations

Definition : An equation in which the reciprocal of every root is also a root, is called a Reciprocal equation

Eg: Consider the equation

$$x^5+x^4+x^3+x^2+x+1=0\dots(1)$$

Replace $x \sum y \frac{1}{n}$,

$$\frac{1}{x^5} + \frac{1}{x^4} + \frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{x} + 1 = 0$$

$$\text{ie } 1+x+x^2+x^3+x^4+x^5=0$$

Which is the original equation

∴ If $x=\alpha$ is a root of (1) then

$x = \frac{1}{\alpha}$ is also a root

Note : The following rules are to be observed before we solve a reciprocal equation.

Rule 1: In a reciprocal equation if the coefficients have all like signs, then -1 is a root of it.

Eg : Consider the equation

$$x^5 + 4x^4 + 3x^3 + 3x^2 + 4x + 1 = 0$$

Here coefficient of $x^5 = \text{const term} = 1$

$$\text{Coeff. of } x^4 = \text{Coeff. of } x = 4$$

$$\text{coeff } x^3 = \text{coeff } x^2 = 3$$

All the coefficients have like signs ie, same sign

Hence By rule 1,

$x = -1$ is a root of it

Rule If the coefficients of the terms equidistant from the first and last term have opposite signs, then $x = 1$ is a root

Eg : Consider

$$6x^5 - x^4 - 43x^3 + 43x^2 + x - 6 = 0$$

Coefficient of $x^5 = 6$, constant term = -6

Coefficient of $x^4 = -1$, coefficient of $x = 1$

Coefficient of $x^3 = -43$, Coefficient of $x = 43$

\therefore The coefficients of the first and last term have opposite signs

$\therefore x = 1$ is a root/

9.2 Examples

(1) Solve $x^4 + 3x^3 - 3x - 1 = 0$

Solution

This is a reciprocal equation of even degree with unlike signs

$\therefore x = 1$ is a root

\therefore By synthetic division.

|

$$\begin{array}{r|rrrrr}
 x=1 & 1 & 3 & 0 & -3 & -1 \\
 & & 1 & 4 & 4 & 1 \\
 \hline
 & 1 & 4 & 4 & 1 & 0
 \end{array}$$

The other roots are given by

$$x^3 + 4x^2 + 4x + 1 = 0$$

This is a reciprocal equation of odd degree with like signs $\therefore x = -1$ is a root

\therefore By synthetic division

$$\begin{array}{r|rrrr}
 x=1 & 1 & 4 & 4 & 1 \\
 & & -1 & -3 & -1 \\
 \hline
 & 1 & 3 & 1 & 0
 \end{array}$$

The other roots are given by $x^2 + 3x + 1 = 0$

$$\begin{aligned}
 x &= \frac{-3 \pm \sqrt{9 - 4}}{2} \\
 &= \frac{-3 \pm \sqrt{5}}{2}
 \end{aligned}$$

Ans : The roots are 1, -1, $= \frac{-3 \pm \sqrt{5}}{2}$

2) Solve $6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0$

This is a reciprocal equation of odd degree with like signs

$\therefore x = -1$ is a root \therefore By synthetic division

$$\begin{array}{r|rrrrrr}
 x=-1 & 6 & 11 & -33 & -33 & 11 & 6 \\
 & & 0 & -6 & -5 & +38 & -5 & -6 \\
 \hline
 & +6 & 5 & -38 & 5 & 6 & 0
 \end{array}$$

The other roots are given by

$$6x^4 + 5x^3 - 38x^2 + 5x + 6 = 0$$

There is a middle term

$$\therefore \div \text{ by } x^2$$

$$6x^2 + 5x - 38 + 5/x + 6/x^2 = 0$$

$$6\left(x + \frac{1}{x}\right) + x\left(\frac{1}{x} + 3\right) - 38 = 0$$

$$6\left[\left(x + \frac{1}{x}\right)\right] + 5x\left(\frac{1}{x} + 3\right) - 38 = 0$$

$$6\left(x + \frac{1}{x}\right)^2 - 2x\left(\frac{1}{x} + 3\right) - 38 = 0$$

$$6\left(x + \frac{1}{x}\right)^2 + x\left(\frac{1}{x} + 3\right) - 38 = 0$$

put $x + \frac{1}{x} = y$

$$\therefore 6y^2 + 5y - 50 = 0$$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 6$$

$$b = 5$$

$$c = -50$$

$$y = \frac{-5 \pm \sqrt{5^2 - 4 \cdot 6 \cdot (-50)}}{2 \cdot 6}$$

$$= \frac{-5 \pm \sqrt{25 + 1200}}{12}$$

$$= \frac{-5 \pm \sqrt{1225}}{12}$$

$$= \frac{-5 + 35}{12}, \frac{-5 - 35}{12}$$

$$y = \frac{+5}{2}, \frac{-40}{12}$$

$$\therefore x + \frac{1}{x} = \frac{5}{2}$$

$$\frac{x^2 + 1}{x} = \frac{5}{2}$$

$$x + \frac{1}{x} = \frac{4}{3}$$

$$\frac{x^2 + 1}{x} = \frac{4}{3}$$

$$2x^2+2=5x$$

$$2x^2-5x+2=0$$

$$x = \frac{5 \pm \sqrt{5-1}}{4}$$

$$x = \frac{5 \pm \sqrt{9}}{4}$$

$$= \frac{5 \pm 3}{4} = 2, \frac{1}{2}$$

$$3x^2+3=-10x$$

$$3x^2+10x+3=-10$$

$$x = \frac{-10 \pm \sqrt{100-36}}{6}$$

$$= \frac{-10 \pm \sqrt{64}}{6}$$

$$= \frac{-10 \pm 8}{6}$$

$$x = \frac{-2}{6}, \frac{-18}{6}$$

$$= \frac{1}{3}, -3$$

∴ The roots are

$$-1, \frac{1}{3}, -3, \frac{1}{2}$$

3) Solve

$$6x^6-25x^5-13x^4-31x^2+25x-6=0$$

Solve

This is a reciprocal equation of even degree of under signs.

∴ x=1 is a root of

By synthetic division

x=1	6	-25	31	0	-31	25	-6
	0	6	-19	+12	12	-19	6
	6	-19	12	+12	-19	6	0

The other roots are given by

$$6x^5-19x^4+12x^3+12x^2-19x+6=0$$

This is a reciprocal equation of odd degree with like signs ∴ x=-1 is a root of it.

∴ By synthetic division

x=-1	6	-19	12	+12	-19	6
------	---	-----	----	-----	-----	---

$$\begin{array}{cccccc|c} 0 & 6 & +25 & -37 & +25 & -6 & \\ \hline 6 & -25 & 37 & -25 & 6 & & 0 \end{array}$$

∴ The other roots are given

$$6x^4 - 25x^3 + 37x^2 - 25x + 6 = 0$$

These is a middle term

Divide by x^2

$$6x^4 - 25x^3 + 37x^2 - 25/x + 6/x = 0$$

$$6x^2 + 2\frac{6}{x^2} - 25\frac{2}{x} - 3\frac{5}{x} + 7 = 0$$

$$6x^2 + 2\frac{6}{x^2} - 25\frac{2}{x} - 3\frac{5}{x} + 7 = 0$$

$$6\left(x^2 \pm \frac{1}{x^2}\right) - 5x\left(\frac{1}{x} \pm 3\right) + 7 = 0$$

$$6\left[\left(x \pm \frac{1}{x}\right)^2\right] - 5\left(x \pm \frac{1}{x} + 3\right) + 7 = 0$$

put $x + 1/x = y$

$$\therefore 6[y^2 - 2] - 25y + 37 = 0$$

$$6y^2 - 12 - 25y + 37 = 0$$

$$6y^2 - 25y + 25 = 0$$

$$y = \frac{25 \pm \sqrt{25^2 - 4 \cdot 6 \cdot 25}}{2 \cdot 6}$$

$$= \frac{25 \pm \sqrt{25}}{12}$$

$$= \frac{25}{12}$$

$$y = \frac{30}{12}, \frac{20}{12}$$

$$y = \frac{5}{2}, \frac{5}{3}$$

$$\therefore x + \frac{1}{x} = \frac{5}{2}$$

$$\therefore x + \frac{1}{x} = \frac{5}{3}$$

$$\frac{x^2 + 1}{x} = \frac{5}{2}$$

$$\frac{x^2 + 1}{x} = \frac{5}{3}$$

$$2x^2 + 2 - 5x = 0$$

$$2x^2 - 5x + 2 = 0$$

$$x = \frac{5 \pm \sqrt{5 - 1}}{4}$$

$$x = \frac{5 \pm \sqrt{5 - 8}}{6}$$

$$= \frac{5 \pm \sqrt{9}}{4}$$

$$= \frac{5 \pm \sqrt{-1}}{6}$$

$$= \frac{5 \pm 3}{4}$$

$$= 2, \frac{1}{2}$$

∴ The solution is

$$\left\{ -1, 1, \frac{1}{2}, \frac{1 \pm i\sqrt{-1}}{2} \right\}$$

9.3 Character and Position of roots – Descarte’s rule to signs

If f(x) is a polynomial with real coefficients with the terms arranged in descending power of x, the number of real positive roots of the equation f(x)=0 does not exceed the number of changes in the sign of the coefficients of f(x) written in a row in the order in which they occur.

Also the number of real negative roots of f(x)=0 does not exceed the number of changes in the sign of the coefficients of f(-x) considered in a similar manner.

9.4 Examples

Ex-1. Show that the equation

$$x^6 - x^5 + 3x^4 + 4x - 1 = 0 \text{ has atleast two imaginary roots}$$

$$\text{pf: } f(x) = x^6 - x^5 + 3x^4 + 4x - 1$$

The series of changes in sign are

$$+ \ - \ + \ \ + \ - \ 1$$

These are three changes of sign in f(x)

The equation cannot have more than three +ve real roots.

Step-2 In $f(x)$

$$\begin{aligned} f(-x) &= (-x)^6 - (-x)^5 + 3(-x)^4 + 4(-x) - 1 \\ &= x^6 + x^5 + 3x^4 - 4x - 1 \end{aligned}$$

The series of charges of sign in $f(-x)$ and

+ + + → - -

This is only one change of sign

∴ $f(x)$ cannot have more than one negative real root.

But the given equation has other two imaginary roots.

2) Find the number of imaginary roots and the number and position of the real roots of $x^5 - x^4 - 4x - 1 = 0$

Charges of sign

+ → - - -

There is only one change of sign.

∴ The no of +ve real roots is ≤ 1

Step 2 : $f(-x) = (-x)^5 - (-x)^4 - 4(-x) - 1$

$$= -x^5 - x^4 + 4x - 1$$

Arranges of sign - - + -

These are two changes of sign in $f(x)$

∴ The number of negative real roots is ≤ 2

The number of real roots is ≤ 3

But the equation is of degree 5 mix

∴ There can be minimum two imaginary roots.

Position:

$$f(x) = x^5 - x^4 - 4x - 1$$

x	2	1	0	-1	-2
f(x)	+	-	-	+	-

∴ The positive real root is between 2 and 1. These are two negative real roots is one root between 0 and -1 and -1 and -2.

9.5 Symmetric Function of roots

By symmetric function of the roots, we mean a function consists of all the roots and which remains unaltered if any two roots are interchanged.

If $\alpha_1, \alpha_2, \alpha_3$ are the roots of the equation

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0, \text{ then}$$

$$\alpha_1 + \alpha_2 + \alpha_3, \alpha_1^2 + \alpha_2^2 + \alpha_3^2, \alpha_1^3 + \alpha_2^3 + \alpha_3^3$$

.....are all symmetric functions of eth roots.

Result-1

Let $\alpha_1 + \alpha_2 + \alpha_3, \dots, \alpha_n$ be the roots of the equation $x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n$
 $x^{n-1} + p_n = 0$ let s_r denote $\alpha_1^r + \alpha_2^r + \alpha_3^r + \dots + \alpha_n^r$

then (i) if $r \leq n$,

$$s_r + s_{r-1}p_1 + s_{r-2}p_2 + \dots + s_1p_{r-1} + rp_r = 0$$

9.6. Examples 1.

If $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$, find the value of

(i) $\sum \frac{1}{\alpha_1}$ (ii) $\sum \frac{1}{\alpha_1 \alpha_2}$ (iii) $\sum \frac{1}{\alpha_1^2}$

Solution : The equation is

$$x^4 + px^3 + qx^2 + rx + s = 0$$

$$a_0 = 1, a_1 = p; a_2 = q; c_3 = r; c_4 = s$$

$$\sum \alpha_1 = \frac{a_1}{a_0} = -p$$

$$\sum \alpha_1 \alpha_2 = \frac{q}{a_0} = q$$

$$\sum \alpha_1 \alpha_2 \alpha_3 = \frac{r}{a_0} = r$$

$$\sum \alpha_1 \alpha_2 \alpha_3 \alpha_4 = \frac{s}{a_0} = s$$

$$\begin{aligned}
 \text{(i)} \quad \sum \frac{1}{\alpha_1} &= \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4} \\
 &= \frac{\alpha_2 \alpha_3 \alpha_4 + \alpha_1 \alpha_3 \alpha_4 + \alpha_1 \alpha_2 \alpha_4 + \alpha_1 \alpha_2 \alpha_3}{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \\
 &= \frac{\sum \alpha_1 \alpha_2 \alpha_3}{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \\
 &= \frac{-r}{s}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \sum \frac{1}{\alpha_1 \alpha_2} &= \frac{1}{\alpha_1 \alpha_2} + \frac{1}{\alpha_1 \alpha_3} + \frac{1}{\alpha_1 \alpha_4} + \frac{1}{\alpha_2 \alpha_3} + \frac{1}{\alpha_2 \alpha_4} + \frac{1}{\alpha_3 \alpha_4} \\
 &= \frac{\alpha_3 \alpha_4 + \alpha_2 \alpha_4 + \alpha_2 \alpha_3 + \alpha_1 \alpha_4 + \alpha_1 \alpha_3 + \alpha_1 \alpha_2}{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \\
 &= \frac{\sum \alpha_1 \alpha_2}{\alpha_1 \alpha_2 \alpha_3 \alpha_4} \\
 &= \frac{q}{s}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad \sum \frac{1}{\alpha_1^2} &= \frac{1}{\alpha_1^2} + \frac{1}{\alpha_2^2} + \frac{1}{\alpha_3^2} + \frac{1}{\alpha_4^2} \\
 &= \left(\sum \frac{1}{\alpha_1} \right)^2 - \sum \frac{1}{\alpha_1 \alpha_2} \\
 &= \frac{r^2}{s^2} - \frac{q}{s} = \frac{r^2 - 2qs}{s^2}
 \end{aligned}$$

Example 2: Find the sum of the fourth powers of the roots of the equation

Solution

$$x^4 - 5x^3 + x - 1 = 0$$

This is of the form

$$x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$$

$$p_1 = -5, p_2 = 0, p_3 = +1, p_4 = -1$$

Let $\alpha, \beta, \gamma, \delta$ be the roots of the given equation

$$\text{To find } \alpha^4 + \beta^4 + \gamma^4 + \delta^4$$

ie. To find s^4 which is of the find sr

$$r=4$$

$$n=4$$

$$\therefore r=4$$

Formula

$$S_r + S_{-1} p_1 + S_{-2} p_2 + \dots + S_{-r+1} p_{r-1} + r p_r = 0$$

Put $r=1$ $s_1 + p_1 = 0$

$$s_1 - 5 = 0$$

$$s_1 = 5 \dots (1)$$

$$r=2$$

$$S_2 + S_1 p_1 + 2p_2 = 0$$

$$S_2 + 5(-5) + 0 = 0$$

$$S_2 = 25$$

put $r=3$

$$S_3 + S_2 p_1 + S_1 p_2 + 2p_2 = 0$$

$$S_3 + 25(-5) + 0 + 3(1) = 0$$

$$S_3 - 125 + 3 = 0$$

$$S_3 - 122 = 0$$

$$S_3 = 122$$

Put $r=4$

$$S_4 + S_3 p_1 + S_2 p_2 + S_1 p_3 + 4p_4 = 0$$

$$S_4 + 122(-5) + 0 + 5(-5) + 4(-1) = 0$$

$$S_4 - 610 + 5 - 4 = 0$$

$$S_4 - 609 = 0$$

$$S_4 = 609$$

Example-3

Find the sum of the sixth powers of the roots of the equation

$$x^4 - x^3 - 7x^2 + x + 6 = 0$$

Solve

$$p_1=-1, p_2=-7, p_3=1, p_4=6$$

let $\alpha \beta \gamma$ be the roots of the given equation.

$$n=4, r=6$$

$$\therefore r > n$$

To find $\alpha^6 + \beta^6 + \gamma^6 + \delta^6$

Formula where $r > n$

$$S_r + S_{-1} p_1 S_{-2} p_2 S_{-3} p_3 \dots S_{-r+1} p_{r-1} = 0$$

$$(2) \text{ when } r \geq n, S_r + S_{-1} p_1 S_{-2} p_2 S_{-3} p_3 \dots S_{-r+1} p_{r-1} = 0$$

$$\text{Put } r=1 \quad s_1 + p_1 = 0$$

$$s_1 - p_1 = 0$$

$$s_2 + 1(-1) + 2(-7) = 0$$

$$s_2 - 15 = 0$$

$$s_2 = 15$$

$$s_3 + s_2 p_1 + s_1 p_2 + 3 p_3 = 0$$

$$s_3 + 15x(-1) + 1x(-7) + 3(1) = 0$$

$$s_3 - 15 - 7 + 3 = 0$$

$$s_3 - 19 = 0$$

$$s_3 = 19$$

$$s_4 + s_3 p_1 + s_2 p_2 + s_1 p_3 + 4 p_4 = 0$$

$$s_4 + 19x(-1) + 15x(-7) + 1x(+1) + 4x6 = 0$$

$$s_4 - 10 - 105 + 1 + 24 = 0$$

$$s_4 - 99 = 0$$

$$s_4 = 99$$

Step 2

Put $r = 5$ in (1)

$$S_5 + S_4p_1 + S_3p_2 + S_2p_3 + S_1p_4 = 0$$

$$S_5 + 99(-1) + 19x(-7) + 15x(+1) + 1x6 = 0$$

$$S_5 - 99 - 133 + 15 + 6 = 0$$

$$S_5 - 21 = 0$$

$$\therefore S_5 = 21$$

$$S_6 + S_5p_1 + S_4p_2 + S_3p_3 + S_2p_4 = 0$$

$$S_6 + 211(-1) + 99(-7) + 19x1 + 15x6 = 0$$

$$S_6 + 217 - 693 + 19 + 90 = 0$$

$$S_6 - 904 + 109 = 0$$

$$S_6 - 795 = 0$$

$$S_6 = 795$$

9.7. Let us sum up

We have so far learnt about solving reciprocal equations, finding the nature of the roots of a given equation.

9.8. check your progress

- (1) Find the nature of the roots of $x^3 - 2x^2 - 3x - 4 = 0$
- (2) Check whether $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$ a reciprocal equation
- (3) Can $x = -1$ be a root of the equation $6x^6 - 25x^3 + 31x^4 - 31x^2 + 25x - 6 = 0$

9.9 Lesson End Activities

- (1) Solve $6x^6 - 25x^3 + 31x^4 - 31x^2 + 25x - 6 = 0$ (Ap 97) Bharathiar
(Ap 2005) Bharathiar
- (2) Show that the sum of 20th powers of the roots of the equation $x^4 + ax + b = 0$ is $50a^4b^2 - 4b^5$ (Ap 97)
- (3) Solve

$$6x^6 - 35x^5 + 56x^4 - 56x^2 + 35x - 6 = 0$$

(4) Prove that the equation

$$x^6 + 3x^2 - 5x + 1 = 0 \text{ has atleast four imaginary roots. (Qp 2005)}$$

9.10. Points for discussion

(1) Show that all the roots of the equation

$$2x^3 - 3x^2 - 12x + 1 = 0 \text{ are real and distinct}$$

(2) If α, β, γ be the roots of $x^3 + px^2 + qx + r = 0$

find the values of

$$(a) \sum \alpha^3 \quad (b) \sum (\alpha + \beta + \gamma)^3 \quad (c) \sum \left(\frac{\beta}{\gamma} + \frac{\gamma}{\beta} \right)$$

9.11. References

A Text Book of Algebra by T.K. Manickavasagam Pillai

UNIT III
Lesson - 10
MULTIPLE ROOTS

Contents

- 10.0. Aims and Objectives
- 10.1. Rules for finding the multiple roots of an equation $f(x)=0$
- 10.2. Example
- 10.3. Let us sum up
- 10.4. Check your progress
- 10.5. Lesson End Activities
- 10.6. Points for discussion
- 10.7. References

10.0 Aims and Objectives

We shall study how to solve an equation whose roots are multiple roots.

Multiple roots

10.1. Rules for finding the multiple roots of an equation $f(x)=0$

- (1) find $f'(x)$
- (2) Find the HCF of $f(x)$ and $f'(x)$
- (3) Find the roots of HCF

10.2 Example

Find the multiple root of the equation

$$x^4 - 9x^2 + 4x + 12 = 0$$

$$f(x) = x^4 - 9x^2 + 4x + 12$$

$$f'(x) = 4x^3 - 18x + 4$$

The HCF of $f(x)$ and $f'(x)$ are $x-2$

$\therefore (x-2)^2$ is a factor of $f(x)$

$$(x-2)^2 = x^2 - 4x + 4$$

$$(x^2 - 4x + 4) \left(x^4 - 9x^2 + 4x + 12 \right) \left(x^2 - 4x + 3 \right)$$
$$\underline{x^4 - 4x^3 + 4x^2}$$

$$\begin{array}{r} 4x^3 - 13x^2 + 4x \\ \underline{4x^3 - 16x + 16x} \\ 3x^2 - 12x + 12 \\ \underline{3x^2 - 12x + 12} \\ 0 \end{array}$$

The other roots of $f(x)=0$ are given by

$$x^2 + 4x + 3 = 0$$

$$(x+3)(x+1) = 0$$

$$x = -3, -1$$

roots of $f(x)=0$ are 2, 2, -1, -3

10.3 Let us sum up

We have studied so far in spring an equation whose multiple roots are given.

10.4. Check your progress

(1) Solve $x^3 - x^2 - 8x + 12 = 0$ has a double root

10.5. Lesson End Activities

(1) Solve

$$6x^6 - 25x^3 + 31x^4 - 31x^2 + 25x - 6 = 0 \quad (\text{Apr 97, Apr 2005})$$

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(2) Show that the sum of the 20th powers of the roots of the equation

Examples

(1) Solve $2x^4 - 12x^3 + 19x^2 - 6x + 9 = 0$ given that it has two equal roots

$$\text{Ans : } 3, 3, \pm \frac{i}{\sqrt{2}}$$

(2) Solve $x^5 - x^4 - 4x^2 + 7x - 3 = 0$ given that it has multiple roots.

$$\text{Ans : } 1, 1, 1, -1 \pm i\sqrt{2}$$

(3) Search for multiple roots and hence solve

$$x^4 - 6x^3 + 13x^2 - 24x + 36 = 0$$

$$\text{Ans : } 3, 3, 2\sqrt{-1}, -\sqrt{-1}$$

10.6. Points for discussion

(4) If the equation $x^4+ax^3+bx^2+x+36=0$ has three equal roots show that each of them is equal to $\frac{6c-a}{3a^2-b}$

10.7. References

A book of Calculus by T.K. Manickavasagam Pillai

Lesson – 11

ROLLE'S THEOREM

Contents

- 11.0 Aims and Objectives
- 11.1. Examples
- 11.2 Let us sum up
- 11.3. Check your progress
- 11.4. Lesson End Activities
- 11.5. Points for discussion
- 11.6. References

11.0 Aims and Objectives

Our aim is to study on Rolle's Theorem.

Rolle's Theorem

Between two consecutive real roots a_1 and a_2 of the equation $f(x)=0$ where $f(x)$ is a polynomial, there lies atleast one real root of the equation $f'(x)=0$

Results :

- (1) If all the roots of $f(x)=0$ are real, then all the roots of $f'(x)=0$ are also real.
- (2) If $f(x)=0$ is a polynomial of degree n , $f'(x)=0$ is a polynomial of degree $(n-1)$ and each root of $f'(x)=0$ lies in each of the $(n-1)$ intervals between the n roots of $f(x)=0$
- (3) If all the roots of $f(x)=0$ are real, then the roots of $f'(x)=0$, $f''(x)=0$ $f'''(x)=0$ are also real
- (4) At most only real root of $f(x)=0$ can lie between two consecutive roots of $f'(x)=0$
- (5) If $f'(x)=0$ has k real roots, then $f(x)=0$ cannot have more than $(k+1)$ roots
- (6) $f(x)=0$ has at least as many imaginary roots as $f'(x)=0$

Position of real roots of $f(x)=0$

11.1. Examples

Find the nature of the roots of the equation $4x^3-21x^2+18x+20=0$

Solution : $f(x) = 4x^3 - 21x^2 + 18x + 20 = 0$

$$\begin{aligned} f'(x) &= 12x^2 - 42x + 18 \\ &= 6[2x^2 - 7x + 3] \\ &= 6[2x^2 - 6x - x + 3] \\ &= 6[2x - (x-3) - 1(x-3)] \\ &= 6[(2x-1)(x-3)] \end{aligned}$$

$f'(x)=0 \rightarrow 2x-1=0, x-3=0$

$x=1/2, x=3$

\therefore The roots of $f'(x)$ are $x=1/2, 3$.



\therefore The roots of $f(x)=0$ if any, will be with internals $(-\alpha, 1/2)$, $(1/2, 3)$ and $(3, \alpha)$

In $(-\alpha, 1/2)$	In $(1/2, 3)$	In $(3, \alpha)$
$f(-\alpha) = -$	$f(1/2) = +$	$f(3) = -$
$f(1/2) = +$	$f(3) = -$	$f(\alpha) = +$

$\therefore f(x)$ has three real roots

Eg.2 Show that the equation $3x^4 - 8x^3 - 6x^2 + 24x - 7 = 0$ has one positive, one negative and two imaginary roots.

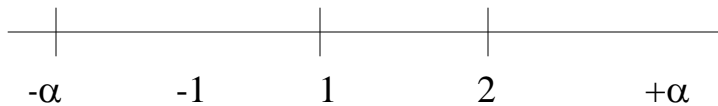
Solution $f(x) = 3x^4 - 8x^3 - 6x^2 + 24x - 7$

$$\begin{aligned} f'(x) &= 12x^3 - 24x^2 - 12x + 24 \\ &= 12[x^3 - 2x^2 - x + 2] \\ &= 12[x^2(x-2) - 1(x-2)] \\ &= 12[(x-2)(x^2-1)] \end{aligned}$$

$f'(x)=0 \rightarrow x=0, x^2-1=0$

$x=2 \quad x=\pm 1$

\therefore The intervals are

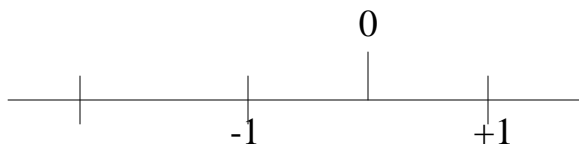


$(-\alpha, -1)$ $(-1, 1)$ $(1, 2)$ $(2, \alpha)$

<p>In $(-\alpha, -1)$</p> <p>$f(-\alpha) = +$</p> <p>$f(-1) = -$</p> <p>In $(2, \alpha)$</p> <p>$f(2) = +$</p> <p>$f(\alpha) = +$</p>	<p>In $(-1, 1)$</p> <p>$f(-1) = -$</p> <p>$f(1) = +$</p> <p>$\therefore x : -\alpha$</p> <p>$f(x) : +$</p>	<p>In $(1, 2)$</p> <p>$f(1) = -$</p> <p>$f(2) = +$</p> <p>1 2 $+\alpha$</p> <p>$+$ $+$ $+$</p>
---	---	---

$f(x)=0$ has a real root lying in $(-\alpha, -1)$ and $(-1, 1)$. The given equation is of 4th degree. \therefore The equation has two imaginary roots.

Step 2 $f(x) = 3x^4 - 8x^3 - 6x^2 + 2x - 7$



$f(0) = -7$

$f(1) = +$

\therefore There is a positive real root in $(-1, 1)$.

Step 3

In $(-\alpha, -1)$

There is a negative real root.

11.2 Let us sum up

We have studied so far how to find the roots using Rolle's Theorem.

11.3. Check your progress

(1) Show that all the roots of $x^3 - 7x + 5 = 0$ are real

11.4. Lesson End Activities

(1) Show that the equation

$x^6 - x^5 + 3x^4 + 4x - 1 = 0$ has atleast two imaginary roots.

11.5. Points for discussion

(2) Find the number of imaginary roots and the number of real roots of the equation

$$x^6 - x^4 - 4x - 1 = 0 \text{ (Answer positive real roots } \leq 1 \text{ no of real roots } \leq 3.$$

Atleast two imaginary roots)

11.6. References

A Text Book of Algebra by T.K. Manickavasagam Pillai

Lesson – 12

NEWTON'S METHOD OF APPROXIMATION TO A ROOT

Contents

12.0 Aims and Objectives

12.1. Newton's method of approximation to a root

12.2 Examples 1:

12.3 Let us sum up

12.4 Check your progress

12.5. Lesson End Activities

12.6 Points for discussion

12.7. References

12.0 Aims and Objectives

We shall find a root by Newton's method.

12.1. Newton's method of approximation to a root

Consider the equation $f(x)=0$ let $x=\alpha$ be a real unrepeated root of $f(x)=0$.

Let $\alpha+h$ be a the actual root, where h is small

By Taylor's Theorem

$$f(\alpha+h)-f(\alpha)+hf'(\alpha)+\frac{h^2}{2}f''(\alpha)+\dots\dots\dots$$

$\therefore x=\alpha+h$ is a root of $f(x)=0$, we have $f(\alpha+h)=0$

$$\therefore f(\alpha)+hf'(\alpha)+\frac{h^2}{2}f''(\alpha)+\dots\dots\dots=0$$

$\therefore h$ is small, neglect h^2 and higher powers of h

$$f(\alpha)+hf'(\alpha)=0$$

$$h = \frac{-f(\alpha)}{f'(\alpha)}$$

$$\therefore \alpha + h \approx \frac{f(\alpha)}{f'(\alpha)}$$

$= \alpha_1$ (say) is a closer

approximation than α

Repeating the process

$$\alpha^2 = \alpha - \frac{f(\alpha_1)}{f'(\alpha_1)} \text{ which is a closer approximation than } \alpha_1$$

continuing this process, the root can be obtained to any desired degree of accuracy.

12.2 Examples 1:

Find the positive real root of $x^3 + 2x^2 + 5x - 220 = 0$ correct to two places of decimals

Solution

$$f(x) = x^3 + 2x^2 + 5x - 220$$

$$f'(x) = 3x^2 + 4x + 5$$

$$f(1) = -ve; f(2) = -ve; f(3) = -ve; f(4) = -ve$$

$$f(5) = -ve; f(6) = +ve$$

There is a root lies between 5 and 6. Take 5 as the appointment value of the root.

$$\alpha = 5$$

$$f(\alpha) = f(5) = -20$$

$$f'(\alpha) = f'(5) = 100$$

The second approximation is

$$\alpha_1 = \alpha - \frac{f(\alpha)}{f'(\alpha)} = 5 - \frac{-20}{100} = 5.2$$

$$\alpha_2 = \alpha_1 - \frac{f(\alpha_1)}{f'(\alpha_1)} = 5.2 - \frac{f(5.2)}{f'(5.2)} = 5.193$$

$$\alpha_3 = 5.193 - \frac{f(5.193)}{f'(5.193)} = 5.1935$$

Coeff. to two places of decimal, the required root = 5.19

12.3 Let us sum up

So far we have studied how to approximate a root of the given equation by Newton's method.

12.4 Check your progress

(1) Find the first approximation of the roots $x^3+x^2+x-100=0$ by Newton's Method.

(2) Find the second approximation of the roots of $5x^3+10x^2-86=0$ by Newton's method.

12.5. Lesson End Activities

(1) By Newton's method of approximation, find to two decimal places, the root of $x^3+x^2+x-100=0$ which is approximately equal to 4.

(2) By Newton's method, find the real root of $5x^3+10x^2-86=0$ correct to three places of decimal (Ans : 2.059)

(3) By Newton's method, find the negative root of $4x^3+x^2+6x+45=0$ correct to three decimal places (-2.096)

12.6 Points for discussion

(4) Find to two places of decimal the positive root of $x^3-3x-4=0$ (Ans 2.20)

(5) Find the positive root of $x^4-8x^2+5x-8=0$ correct to two places of decimals.

12.7. References

A Text Book of Algebra by T.K. Manickavasagam Pillai

Lesson – 13

HORNER'S METHOD

Contents

- 13.0. Aims and Objectives
- 13.1. Horner's Method
- 13.2. Examples
- 13.3 Let us sum up
- 13.4 Check your progress
- 13.5 Lesson End Activities
- 13.6. Points for discussion
- 13.7. References

13.0 Aims and Objectives

We shall study the solving of an equation by Horner's Method.

13.1 Horner's Method

We shall consider the method of finding an approximate value of a real root of the equation $f(x)=0$ bying between two positive integers when there is only one root between those integers.

13.2. Examples

Find the positive root of the equation $x^3+x^2-7x-3=0$ correct to two decimal places by Horner's method.

Step – 1 $f(x) = x^3+x^2-7x-3$

$f(2) = -ve$

$f(3) = +ve$

There is a positive root bying between 2 and 3

By synthetic division

x=2	1	1	-7	-3	
	0	2	6	-2	
x=2	1	3	-1	-5	
	0	2	10	-5	
	1	5	9	-5	

$$\begin{array}{r|l}
 x=2 & \begin{array}{cc} 0 & 2 \\ \hline 1 & 7 \end{array}
 \end{array}$$

∴ The transformed equations

$$x^3 + 7x^2 + 9x - 5 = 0$$

Equation (2) has a root lies between 0 and 1 to find first decimal
multiply the roots of (2) by 10

$$x^3 + 70x^2 + 900x - 5000 = 0$$

Hint In Equation (2) Coeff $fx \times 10$, Coeff $fx^2 \times 100$, Constant term $\times 1000$

$$x^3 + 70x^2 + 900x - 5000 = 0$$

$$f_1(x) = x^3 + 70x^2 + 900x - 5000$$

$$f_1(4) = -ve$$

$$f_1(5) = +ve$$

∴ A root of $f_1(x) = 0$ lies between 4 and 5

step 2 Diminish the roots of $f_1(x) = 0$ by 4

$$\begin{array}{r|l}
 x=4 & \begin{array}{cccc} 1 & 70 & 900 & -5000 \\ \hline & 4 & 296 & 4784 \end{array} \\
 x=4 & \begin{array}{ccc|c} 1 & 74 & 1196 & -216 \\ \hline 0 & 4 & 312 & \end{array} \\
 x=4 & \begin{array}{cc|c} 14 & 78 & 1508 \\ \hline 0 & 4 & \end{array} \\
 & \begin{array}{r|l} 1 & 82 \end{array}
 \end{array}$$

∴ The transformed equation

$$x^3 + 82x^2 + 1508x - 216 = 0 \text{ ---(3)}$$

There is a root lying between 0 and 1 to find second decimal

Multiply the roots by 10

$$x^3 + 820x^2 + 150800x - 216000 = 0 \text{ ---(4)}$$

$$f_2(x) = x^3 + 820x^2 + 150800x - 216000$$

$$f_2(1) = -ve$$

$$f_2(2) = +ve$$

The root lies between 1 and 2

\therefore The required root is 2.41

13.3 Let us sum up

So far we have studied the concept of solving an equation using Horner's method.

13.4 Check your progress

(1) Find a positive root of the equation $x^3 + x^2 - 7x - 3 = 0$

(2) Find a positive root of the equation $x^3 - 5x - 11 = 0$

13.5 Lesson End Activities

(1) Using Horner's method, find a positive root of the equation $x^3 - 3x + 1 = 0$ correct to three places of decimals.

(Apr 1984, Apr 2003,

Nov 2005, Nov 2004)

(Ans : 1.532)

(2) Find the positive root of the equation $x^3 - 2x^2 - 3x - 4 = 0$ correct to three places of decimals (Ans 3.284)

(Nov 2006

Bharathiar)

(3) Show that the equation $x^3 - 6x - 13 = 0$ has one real root between 3 and 4 and find it to two places of decimals.

(Ans: 3.18)

(4) Find by Horner's method the root of the equation $x^3 + x^2 - 2x - 1 = 0$ that lies between 1 and 2

(Apr 2005)

(5) Find the positive root of the equation $x^3 - 2x^2 - 3x - 4 = 0$ correct to three places of decimals using Horner's method (Apr 2005, Apr 2004, Apr 1983, Nov 2000, Bharathiar)

13.6 Points for discussion

(1) Using Horner's method find the root of $x^3 + 9x^2 - 18 = 0$ which lies between 1 and 2 correct to two decimal places (Apr 2005, Bharathiar)

(2) Find by Horner's method, the positive root of the equation $x^3 - 4x^2 + 5 = 0$ correct to two significant places of decimals (Nov 2003, Bharathiar)

13.7 References

A Text Book of Algebra by T.K. Manickavasagam Pillai

Lesson-14

RADIUS OF CURVATURE

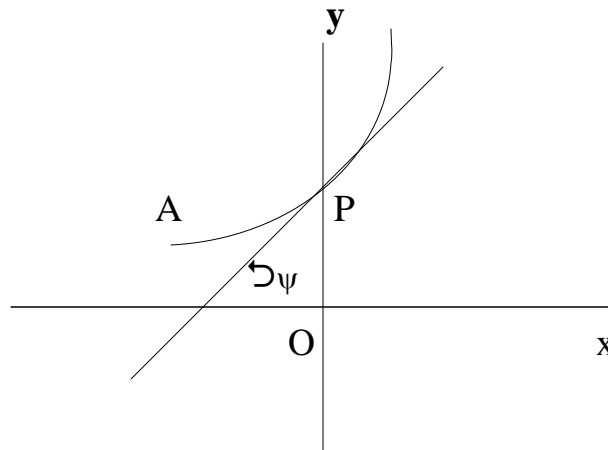
Contents

- 14.0 Aims and Objectives
- 14.1 Curvature in Cartesian form
 - 14.1.1 Radius of Curvature
- 14.2. Radius of Curates in Cartesian
- 14.3. Examples
- 14.4 Let us sum up
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- 14.6. Lesson End Activities
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14.0 Aims and Objectives

Our Aim is to learn the concept of finding the radius of Curvature in Cartesian Coordinates.

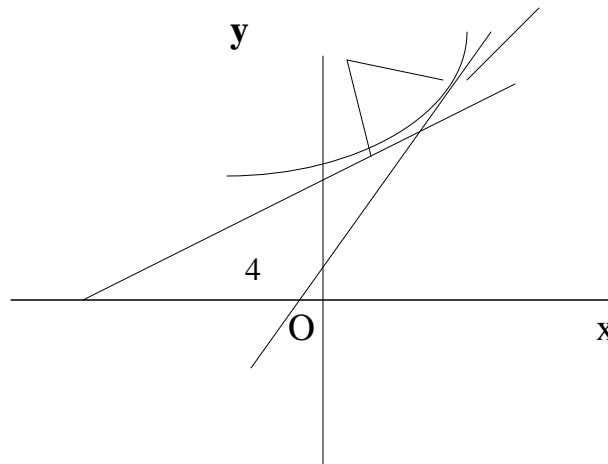
14.1 Curvature in Cartesian form



Let A be a fixed point on the curve $y=f(x)$ let s denote the arc along AP measured from A in the curve $y=f(x)$ where P is a any point on $y=f(x)$ let the tangent at P make an angle ψ with the x axis. Then, as P moves along the curve, s and ψ vary and the rate of which ψ increases relative to s is called the curvature of the curve as ρ i.e. in symbols $d\psi/ds$ is called the curvature

of the curve at p . In other words the curvature is the rate of change of the direction of the tangent at P. Also curvature of a circle is the reciprocal of its radius.

14.1.1 Radius of Curvature



Consider the curve $y=f(x)$

Let P and Q be two points on this curve. Let the tangents at P and Q make angles ψ and $\psi + d\psi$ respectively with x axis let the normals at P and Q intersect C as let A be a fixed point on the curve. Let arc AP=s

$$\text{Arc AP} = s \quad PQ = \Delta s$$

As Q tends to P, limit of PC is $\frac{ds}{d\psi}$

$$\therefore PC = \frac{ds}{d\psi}$$

$$\therefore \frac{1}{PC} = \frac{d\psi}{ds}$$

$\frac{1}{PC} = \frac{d\psi}{ds}$ is called the radius of curvature at p.

The circle with the centre at C and radius=PC has the same tangent and the same curvature as the curve $y=f(x)$ has at P.

14.2. Radius of Curvature in Cartesian

Consider the curve $y=f(x)$. Let A be a fixed point on it. Let P be any point on it.

Let the tangent at p make an angle ψ with the x axis

then $dy/dx = \tan \psi$

differentiating with respect to x

$$\frac{d^2 y}{dx^2} = \frac{y}{s} e^{2\psi} \frac{d\psi}{dx}$$

$$= s e^{2\psi} \frac{d\psi}{s dx}$$

But $\frac{dx}{ds} = c \cot \psi$

$$\therefore \frac{d^2 y}{dx^2} = \frac{y}{s} e^{2\psi} \frac{d\psi}{dx} \cdot \frac{1}{c \cot \psi}$$

$$= s e^{3\psi} \frac{d\psi}{dx}$$

$$\frac{dx}{d\psi} = \frac{s e^{3\psi}}{\frac{d^2 y}{dx^2}}$$

$$\rho = \frac{(s e^3)^{3/2}}{\frac{d^2 y}{dx^2}}$$

where $\rho = \frac{dx}{d\psi}$, the radius of curvature.

$$= \frac{(1 + \tan^2 \psi)^{3/2}}{\frac{d^2 y}{dx^2}}$$

$$= \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2 y}{dx^2}}$$

$$\therefore \rho = \frac{(1 + y_1^2)^{3/2}}{y_2}$$

where $y_1 = \frac{dy}{dx}$
 $y_2 = \frac{d^2 y}{dx^2}$

14.3. Examples

(1) Find the radius of curvature of the curve $xy=30$ at $(3,10)$

Solution $xy=30$ ----(1)

diff 10.r.t.x

$$x \frac{d}{dx} y = -y \dots(2)$$

$$\frac{d}{dx} \frac{y}{x} = \frac{y}{x^2}$$

$$\left. \frac{d}{dx} \right\} (3,10) = \frac{1}{3}$$

$$= \frac{2}{9}$$

$$y_1 = \frac{1}{3}, \quad \frac{0}{9} = \frac{2}{9}$$

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{\left(1+\frac{1}{9}\right)^{3/2}}{\frac{2}{9}}$$

$$= \frac{(10)^{3/2}}{2} \times \frac{9}{2}$$

$$\rho = \frac{(10)^{3/2}}{6}$$

(2) Find ρ at the point where $x = \pi/2$ to the curve

$$y = 4 \sin x - \sin 2x$$

$$y = 4 \sin x - \sin 2x$$

put $x = \pi/2$

$$y = 4 \sin \pi/2 - \sin 2 \times \pi/2$$

$$= 4 \times 1 - \sin \pi = 4 - 0 = 4$$

$$u = 4 \quad \text{at } x = \pi/2$$

\therefore The point is $(\pi/2, 4)$

$$y_1 = \frac{d}{dx} (4 \cos 2x) = -8 \cos 2x$$

$$y_2 = \frac{d^2 y}{d^2 x} \text{ at } \theta = \frac{\pi}{4}$$

$$\begin{aligned} y_1) \left(\frac{\pi}{2}, 4 \right) &= 2 \cos \frac{\pi}{2} = 2 \cdot 0 = 0 \\ &= 2 \cos \theta = 2 \cos \frac{\pi}{2} \\ &= 2 \end{aligned}$$

$$\begin{aligned} y_2) \left(\frac{\pi}{2}, 4 \right) &= 2 \sin \frac{\pi}{2} = 2 \cdot 1 = 2 \\ &= -4 \end{aligned}$$

$$\therefore \boxed{y_1 = 2} \quad \boxed{y_2 = -4}$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{(1 + 4)^{3/2}}{-4} = \frac{5^{3/2}}{-4}$$

In magnitude $\rho = \frac{5^{3/2}}{4}$

$$= \frac{\sqrt[5]{5}}{4}$$

(3) Find the radius of curvature of the curve at $x = a \cos \theta$, $y = b \sin \theta$ at

$$\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}} \right)$$

If: Step 1

$$x = a \cos \theta \quad y = b \sin \theta$$

$$\frac{d^2 x}{d^2 \theta} = -a \sin \theta \quad \frac{d^2 y}{d^2 \theta} = b \cos \theta$$

$$\frac{d^2 y}{d^2 x} = \frac{b \cos \theta}{-a \sin \theta}$$

$$\begin{aligned} \frac{d^2 y}{d^2 x} &= \frac{y}{x} \cdot \frac{d^2 y}{d^2 \theta} \cdot \frac{d^2 \theta}{d^2 x} = \frac{b}{a} \cos \theta \cdot \frac{1}{-\sin \theta} \\ &= -\frac{b}{a} \cot \theta \end{aligned}$$

$$x = \frac{a}{\sqrt{2}} \cos \theta \quad \theta = \frac{\pi}{4}$$

$$y = \frac{b}{\sqrt{2}} \sin \theta \quad \theta = \frac{\pi}{4}$$

$$\therefore A = \frac{\pi}{4} \Rightarrow y = c - \frac{b}{a} = \frac{b}{a} - \frac{b}{c}$$

$$y^2 = \frac{b}{a^2} (c - \frac{b}{a})^2 = \frac{b}{a^2} \left[c^2 - \frac{2bc}{a} + \frac{b^2}{a^2} \right]$$

$$= \frac{b}{a^2} x (\sqrt{2})^3 = 2 \frac{b}{a^2} \sqrt{2}$$

$$\rho = \frac{(1 + y_1^2)^{3/2}}{y_2} = \frac{\left[1 + \frac{b^2}{a^2} \right]^{3/2}}{-\frac{b}{a^2} \times 2\sqrt{2}}$$

$$= \frac{-a^2 (a^2 + b^2)^{3/2}}{b a^3} \cdot \frac{1}{2\sqrt{2}}$$

$$\rho = \frac{1}{2\sqrt{2}} \left(\frac{a^2 + b^2}{b} \right)^{3/2}$$

\therefore In magnitude $\rho = \frac{1}{2\sqrt{2}} \left(\frac{a^2 + b^2}{b} \right)^{3/2}$

(4) Prove that the radius of curvature at a point $(a \cos^3 v, a \sin^3 v)$ on the curve $x^{2/3} + y^{2/3} = a^{2/3}$ is $3 a \sin v \cos v$

Proof $x = a \cos^3 v, y = a \sin^3 v$ is the parametric equation of the curve $x^{2/3} + y^{2/3} = a^{2/3}$.

Solution

$$x = a \cos^3 v$$

$$\frac{d}{d v} \frac{x}{v} = 3ac \cos^2 v \sin v$$

$$y = a \sin^3 v$$

$$\frac{d}{d v} \frac{y}{v} = 3as \sin^2 v \cos v$$

$$\frac{d}{d x} \frac{y}{x} = \frac{\frac{d}{d v} \frac{y}{v}}{\frac{d}{d v} \frac{x}{v}}$$

$$= \frac{3as \sin^2 v \cos v}{-3ac \cos^2 v \sin v}$$

$$\frac{d}{d x} \frac{y}{x} = -\frac{a \sin v}{a \cos v}$$

$$\begin{aligned} & \frac{d^2}{dx^2} \frac{y}{x} = \frac{d}{dx} \left(\frac{dy}{dx} \cdot \frac{1}{x} \right) \\ & = \frac{dy}{dx} \cdot \left(-\frac{1}{x^2} \right) - \frac{1}{x^2} \cdot \frac{dy}{dx} \\ & = -\frac{2}{x^3} \frac{dy}{dx} \\ & y_2 = \frac{1}{3as^2 \cos^3 \theta} \\ p &= \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+\frac{4a^2}{x^2})^{3/2}}{-\frac{2}{x^3} \frac{dy}{dx}} \\ &= \frac{(x^2+4a^2)^{3/2}}{2x^3 \frac{dy}{dx}} \\ p &= 3a \sin \theta \cos \theta \end{aligned}$$

14.4 Let us sum up

We have studied the concept of finding the radius of curvature whenever the equation of a curve is given in the form $y=f(x)$ or in a parametric form i.e. $x=f(t)$; $y=p(t)$.

14.5. Check your progress

- (1) Find the radius of curvature of $y^2=4ax$ at $(1,1)$
- (2) find the radius of curvature of $y=e^x$ at $(0,1)$

14.6. Lesson End Activities

(1) Find the radius of curvatures for the following

Ans

(a) $y=e^x$ at $(0,1)$; $(2\sqrt{2})$

(b) $y = \frac{1}{a} \log \frac{x}{a} = x - 1$ $\left(\frac{2\sqrt{2}}{3} \right)$

(c) $y^3=x(x+2y)$ at $(1,-1)$; $(1/2)$

(d) $4ay^2=(2a-x)^3$ at $(a,a/2)$; $\left(\frac{1-2a^2}{4} \right)$

(2) Find the radius of curvature for the curve $x=a(\cos t + \sin t)$; $y=a(\cos t - \sin t)$

(3) Find the radius of curvature of the curve $y^2=4ax$ at $(at^2, 2at)$

14.7. Points for discussion

- (1) Prove that the radius of curvature of the curve $x=a(\theta+\sin\theta)$; $y=a(1-\cos\theta)$ is $4a\cos\theta/2$
- (2) Find the radius of curvature of the curve $x=f(\theta)$ and $y=\phi(\theta)$

14.8. References

A book of Calculus by T.K. Manickavasagam Pillai

Lesson-15

RADIUS OF CURVATURE IN POLAR FORM

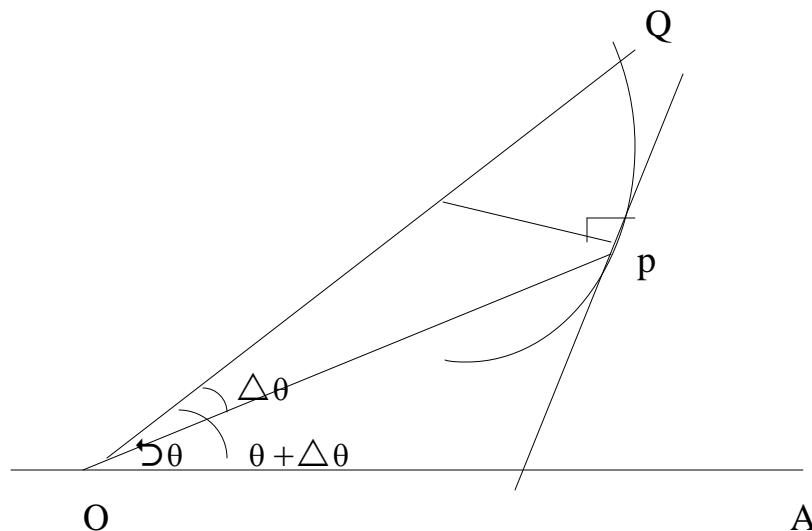
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15.0 Aims and Objectives

Our Aim is to learn the method of finding the radius curvature of a curve given in polar form, the p-r equation of a curve, evolutes and involutes of a given curve.

Radius of curvature in polar forms Bookwork find the radius of curvature of the curve $r=f(\theta)$.



Let $r=f(\theta)$ be the equation of the curve let ρ be any point on it.

Let $OP=r$. Let OP make an angle θ with the initial line OA .

Let the tangent at P make an angle ψ with the initial line

Let Q be a neighboring point of P. on the curve such that $\angle POQ = \theta + \Delta\theta$

$\angle POQ = \Delta\theta$; let $OP = r + \Delta r$

from figure, $\psi = \theta + \phi$

$$\frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta} \quad (1)$$

We have

$$\begin{aligned} \tan \phi &= r \frac{d\phi}{dr} \\ &= r \cdot \frac{1}{\left(\frac{dr}{d\theta}\right)} \end{aligned}$$

$$\tan \phi = \frac{r}{\frac{dr}{d\theta}}$$

differentiating w.r.t. θ

$$\sec^2 \phi \frac{d\phi}{d\theta} = \frac{\frac{dr}{d\theta} \cdot \frac{dr}{d\theta} - r \frac{d^2 r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

$$\sec^2 \phi \frac{d\phi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2 r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

$$\frac{d\phi}{d\theta} = \frac{1}{\sec^2 \phi} \cdot \frac{\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2 r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

$$= \frac{1}{1 + \tan^2 \phi} \cdot \frac{\left(\frac{dr}{d\theta}\right)^2 - r \frac{d^2 r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

$$= \frac{1}{1 + \left(\frac{1}{\frac{d}{d\theta} r} \right)^2} \cdot \frac{\left(\frac{d}{d\theta} r \right)^2 - r \frac{d^2 r}{d\theta^2}}{\left(\frac{d}{d\theta} r \right)^2}$$

$$\frac{d}{d\theta} r = \frac{\left(\frac{d}{d\theta} r \right)^2 - r \frac{d^2 r}{d\theta^2}}{r^2 + \left(\frac{d}{d\theta} r \right)^2}$$

using in (1)

$$\frac{d^4}{d\theta} = 1 + \frac{\left(\frac{d}{d\theta} r \right)^2 - r \frac{d^2 r}{d\theta^2}}{\left(\frac{d}{d\theta} r \right)^2 + r^2}$$

$$\frac{d^4}{d\theta} = \frac{r^2 + 2 \left(\frac{d}{d\theta} r \right)^2 - r \frac{d^2 r}{d\theta^2}}{r^2 + \left(\frac{d}{d\theta} r \right)^2}$$

But we know that

$$\frac{d}{d\theta} = s \sqrt{r^2 + \left(\frac{d}{d\theta} r \right)^2}$$

$$\therefore p = \frac{d}{d\theta} s = \frac{d}{d\theta} \text{ (incomplete)}$$

$$l = \frac{\sqrt{r^2 + \left(\frac{d}{d\theta} r \right)^2} \left[\frac{d^2 r}{d\theta^2} \right]}{r^2 + 2 \left(\frac{d}{d\theta} r \right)^2 - r \frac{d^2 r}{d\theta^2}}$$

$$l = \frac{\left[r^2 + \left(\frac{d}{d\theta} r \right)^2 \right]^{\frac{3}{2}}}{r^2 + 2 \left(\frac{d}{d\theta} r \right)^2 - r \frac{d^2 r}{d\theta^2}}$$

15.1. Example

(1) Find the radius of curvature of the curve $r^2 = a^2 \sec^2 2\theta$

Solution

$$r^2 = a^2 \sec^2 2$$

$$\log r^2 = \log a^2 + \log \sec^2 2$$

$$2 \log r = \log a^2 + \log \sec^2 2$$

$$\therefore \frac{1}{r} \cdot \frac{dr}{d\theta} = \frac{1}{a} \cdot \frac{1}{\sec^2 2} \cdot 2 \tan 2 \sec 2$$

$$\frac{1}{r} \frac{dr}{d\theta} = \tan 2 \sec 2$$

$$\frac{dr}{d\theta} = r \tan 2 \sec 2$$

$$\frac{d^2 r}{d\theta^2} = 2r \sec^2 2 \tan 2 + r \cdot 2 \sec 2 \tan 2$$

$$= 2r \sec^2 2 \tan 2 + 2r \sec 2 \tan 2$$

$$= 2r \sec^2 2 \tan 2$$

$$\therefore r \frac{d^2 r}{d\theta^2} = 2r^2 \sec^2 2 \tan 2$$

$$= 2r^2 \cdot \frac{r^4}{a^4} \sec^2 2 \tan 2$$

$$= \frac{2r^6}{a^4} + r^2 \left(\frac{2}{a} \right)$$

$$= \frac{2r^6}{a^4} + \frac{r^6}{a^4} \cdot \frac{3}{2} \cdot \frac{r^2}{a}$$

$$r^2 + \left(\frac{dr}{d\theta} \right)^2 = r^2 \sec^2 2$$

$$= r^2 (1 + \tan^2 2)$$

$$= r^2 \sec^2 2$$

$$= r^2 \cdot \frac{\theta^4}{a^4} = \frac{\theta^6}{a^4}$$

$$\therefore \left[r^2 + \left(\frac{dr}{d\theta} \right)^2 \right] = \frac{r^2}{a^4} \left(\frac{6r}{4} \right)^2 = \frac{9r^2}{a^6}$$

$$r^2 + 2 \cdot r^2 \cdot \dot{r} \cdot \dot{\theta} = a^2 \sin 2\theta \cdot \frac{3r^6}{a^4} \dot{r}$$

$$= r^2 \cdot 2 \cdot \left(\frac{3r^6}{a^4} \right) \dot{r}$$

$$= 2r^2 + 2 \left[\frac{r^4}{a^4} \right] \frac{3}{a}$$

$$= 2r^2 + 2 \frac{r^6}{a^4} \cdot \frac{3}{a}$$

$$= \frac{r^6}{a^4}$$

$$\therefore p = \frac{r^9}{a^6} \left(-\frac{r^6}{a^4} \right) \frac{3}{a} \frac{4}{r}$$

$$= \frac{r^3}{a^2}$$

$$= \frac{r^2 \dot{r}}{a^2}$$

$$= \frac{a^2 \sin^2 \theta}{a^2}$$

$$= -4 \sec^2 2$$

\therefore In magnitude, $p = r \sec^2 2$

$$r = a^2$$

$$\dot{r} = a$$

$$\dot{\theta} = 0$$

$$p = \frac{[r^2 + \dot{r}^2]^{\frac{3}{2}}}{r^2 + \dot{r}^2 - \dot{\theta}^2 r}$$

$$= \frac{[a^2 \theta^2 + a]^2 \frac{3}{2}}{a^2 \theta^2 + a^2 - 0}$$

$$= \frac{[a^2 (\theta^2 + 1)]^{\frac{3}{2}}}{a^2 (\theta^2 + 1)}$$

$$p = \frac{a^3 (\theta^2 + 1)^{\frac{3}{2}}}{a^2 (\theta^2 + 1)}$$

$$= \frac{a (\theta^2 + 1)^{\frac{3}{2}}}{(\theta^2 + 1)}$$

15.2. Evolutes and Involutives

Definition 1 : The coordinates of the centre of curvature of the curve $y=f(x)$ at any point $p(x,y)$ is given by

$$x = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$y = y + \frac{1+y_1^2}{y_2}$$

Definition 2 : The locus of the centre of curvature of a curve is called the evolute of the curve.

Definition 3 : If the evolutes of a curve is itself, then it is called an involute.

15.3. Examples:

(1) Find the evolutes of the curve $y^2=4ax$ at $(at^2, 2at)$

Solution

$$x=at^2 \quad ; \quad y=2at$$

$$\frac{d}{dt} \frac{x}{t} = 2a \quad ; \quad \frac{d}{dt} \frac{y}{t} = 2a$$

$$y_1 = \frac{d}{dt} \frac{y}{x} \frac{d}{dt} = \frac{2a}{2a}$$

$$= \frac{1}{t}$$

$$y_2 = \frac{d^2}{dt^2} \frac{y}{x} - \frac{d}{dt} \frac{t}{2x} \cdot \frac{1}{t} \cdot \frac{1}{a2} \cdot \frac{1}{t}$$

$$= \frac{1}{2a^3 t}$$

$$x = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$= a^2 t \frac{\frac{1}{t} \left[1 + \frac{1}{t^2} \right]}{-\frac{1}{2a^3 t}}$$

$$= a^2 t \cdot 2t^3 \cdot \frac{1}{t} \left(\frac{t^2 + 1}{t^2} \right)$$

$$\begin{aligned}
 &= a^2 - 2at + t^2 \\
 &= a^2 - 2a^2 + 2a^2 - t^2 \\
 x &= 3a^2 - 2a^2 + a^2 - t^2 \dots (1)
 \end{aligned}$$

step-2

$$\begin{aligned}
 y &= y + \frac{1+y_1^2}{y^2} \\
 &= 2a + t - \frac{1+\frac{1}{t^2}}{-\frac{1}{2}a^3} \\
 &= 2a - t + 2a^3 \frac{(t^2+1)}{t^2} \\
 &= 2at - 2at(1+t^2) \\
 &= 2at - 2at - 2at^3 \\
 &= -2at^3 \dots (2)
 \end{aligned}$$

Eliminate t between (1) and (2)

from (1) $x - 2a = 3at^2$

$$\begin{aligned}
 \frac{x-2a}{3a} &= t^2 \\
 \therefore \left(\frac{x-2a}{3a}\right)^3 &= t^6 \\
 \frac{(x-2a)^3}{2a^3} &= t^6 \dots (3)
 \end{aligned}$$

from (2) $y = -2at^3$

$$\begin{aligned}
 \frac{y}{2a} &= -t^3 \\
 \therefore \left(\frac{y}{2a}\right)^2 &= t^6 \\
 \frac{y^2}{4a^2} &= t^6 \dots (4)
 \end{aligned}$$

from (3) and (4)

$$\frac{(x-2a)^3}{2a^3} = \frac{y^2}{4a^2}$$

$$\therefore 4(x-2a)^3 = 27ay^2$$

\therefore The locus of (x,y) is

Which is the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Solution: Step-1: Any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } (a \cos \theta, b \sin \theta)$$

$$\therefore x = a \cos \theta \quad y = b \sin \theta$$

$$\frac{d}{d\theta} x = -a \sin \theta \quad \frac{d}{d\theta} y = b \cos \theta$$

$$y_1 = \frac{d}{d\theta} \left(\frac{d}{d\theta} y \right) = \frac{d}{d\theta} (b \cos \theta) = -b \sin \theta$$

$$= -\frac{b}{a} c \cos \theta$$

$$y_2 = \frac{d}{d\theta} (b \cos \theta) \frac{d}{d\theta} \theta = -b \sin \theta$$

$$= -\frac{b}{a^2} c \cos \theta$$

step-2

$$X = x^2 - \frac{y_1^2}{2}$$

$$= a^2 \cos^2 \theta - \frac{(-b/a \cos \theta)^2}{2} = a^2 \cos^2 \theta - \frac{b^2 \cos^2 \theta}{2a^2}$$

$$= a^2 \cos^2 \theta - \frac{b^2 \cos^2 \theta}{2a^2} = \cos^2 \theta \left[a^2 - \frac{b^2}{2a^2} \right]$$

$$= a^2 \cos^2 \theta - \frac{b^2 \cos^2 \theta}{2a^2} = \cos^2 \theta \left[a^2 - \frac{b^2}{2a^2} \right]$$

$$= a^2 \cos^2 \theta - \frac{b^2 \cos^2 \theta}{2a^2} = \cos^2 \theta \left[a^2 - \frac{b^2}{2a^2} \right]$$

$$= ac \theta - \frac{1}{a} c \theta [a^2 \theta + b^2 s] c \theta^2 \theta s$$

$$X = \frac{1}{a} [a^2 c \theta^2 - c^2 [a^2 \theta + b^2 s] c \theta^2 \theta s]$$

$$ax = a^2 c \theta^2 - c^2 \theta + a s^3 c \theta - b s c^3 \theta s$$

$$= (a^2 - b^2) c^3 \theta$$

$$\frac{a x}{a^2 - b^2} = c^3 \theta$$

$$\therefore c^3 \theta = \frac{a x}{a^2 - b^2} \dots\dots\dots(1)$$

Step 3

$$Y = y + \frac{1 + y_1^2}{y_2}$$

$$= b s i \theta + \frac{[1 + \frac{b^2}{a^2} \theta]^2 \theta}{-\frac{b}{a^2} c \theta s^3 \theta}$$

$$= b s \theta - \frac{a^2 s \theta^3 + [a^2 \theta + b^2 s] \theta}{b [a^2 s \theta^3 - b s^3 \theta]}$$

$$= b s i \theta - \frac{a^2 s \theta^3 + [a^2 \theta + b^2 s] \theta}{b [a^2 s \theta^3 - b s^3 \theta]}$$

$$= \frac{1}{b} [b^2 s \theta^3 + a^2 \theta] s i \theta (a^2 s \theta^3 - b s^3 \theta)$$

$$Y = \frac{1}{b} [b^2 s \theta^3 + a^2 \theta] s i \theta (a^2 s \theta^3 - b s^3 \theta)$$

$$bY = b^2 s \theta^3 + a^2 \theta - b^2 s \theta^3 - a^2 \theta$$

$$= (b^2 - a^2) s i^3 \theta$$

$$\frac{b Y}{b^2 - a^2} = s i^3 \theta$$

$$\therefore s i^3 \theta = \frac{-b Y}{a^2 - b^2} \dots\dots\dots(2)$$

Step 4

We have $\frac{a x}{a^2 - b^2} = c^3 \theta$

$$\frac{b}{a^2 - b^2} = s \cdot i^3 \theta$$

But $c \cdot \theta = n$

$$\therefore \left(\frac{a}{a^2 - b^2} \right)^{2/3} + \left(\frac{b}{a^2 - b^2} \right)^{2/3} = 1$$

$$\left(\frac{a}{a^2 - b^2} \right)^{2/3} + \left(\frac{b}{a^2 - b^2} \right)^{2/3} = 1$$

$$\left(\frac{a}{a^2 - b^2} \right)^{2/3} + \left(\frac{b}{a^2 - b^2} \right)^{2/3} = 1$$

$$\frac{(a)^{2/3} x}{(a^2 - b^2)^{2/3}} + \frac{b^2 (y)^{2/3}}{(a^2 - b^2)^{2/3}} = 1$$

$$\therefore (a)^{2/3} x + (b^2)^{2/3} y = (a^2 - b^2)^{2/3}$$

\therefore The locus of (x,y) is

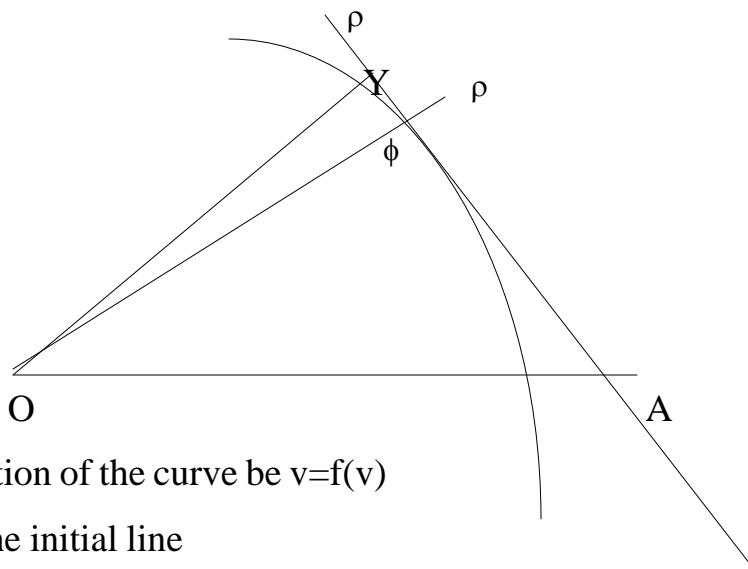
$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

Which is the evolute of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

15.4. Pedal Equation

Find the pedal equation of the curve (p-r equation) $r=f(v)$

proof



let the equation of the curve be $r=f(v)$

let O A be the initial line

let O be the pole

let p be any pt in the curve

Join op let $op=r$ Draw AY the tangent at p .

Let ϕ be the angle between the radius vector op and the tangent at the point p .

in $\triangle opy$

$$\sin(\phi) = \frac{oy}{op}$$

$$\sin\phi = \frac{oy}{r}$$

$$\therefore oy = r \sin\phi$$

$$\therefore p = r \sin\phi$$

$$\frac{1}{p} = \frac{1}{r \sin\phi}$$

$$\frac{1}{p^2} = \frac{1}{r^2 \sin^2\phi} = \frac{1}{r^2} \csc^2\phi$$

$$\frac{1}{p^2} = \frac{1}{r^2} (\sec^2\phi) \quad (1)$$

But $\tan\phi = \frac{1}{r} \frac{d}{\theta}$

$$\cot\phi = \frac{1}{r} \frac{d}{\theta}$$

using no...(1)

$$\frac{1}{p^2} = \frac{1}{r^2} \left(\frac{d}{\theta} \right)^2$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{\theta^2} \left(\frac{d}{r} \right)^2 \dots (2)$$

let $r = \frac{1}{u} \therefore \frac{d}{\theta} = \frac{1}{u} \frac{d}{\theta} \frac{r}{p}$

\therefore using in (2)

$$\frac{1}{p^2} = u^2 + \frac{1}{\theta^2} \left(\frac{d}{u} \right)^2 \left(\frac{r}{p} \right)^2$$

$$\frac{1}{p^2} = u^2 + \left(\frac{d}{\theta} \right)^2$$

15.5 Examples 1:

Find the p-r equation of the cardioid $r=a(1-\cos 2\theta)$

Solution

$$r=a(1-\cos 2\theta)$$

The p-request ion is $\frac{d}{d\theta} \frac{r}{\rho^2} = a \sin \theta$

$$\begin{aligned} \frac{1}{\rho^2} &= \frac{1}{r^2} + \frac{1}{r} \left(\frac{dr}{d\theta} \right)^2 \\ &= \frac{1}{r^2} + \frac{1}{r} a^2 \sin^2 \theta \\ &= \frac{r^2 + a^2 \sin^2 \theta}{r^4} \\ \frac{1}{\rho^2} &= \frac{a^2 (1 - \cos 2\theta + \sin^2 \theta)}{r^4} \\ &= \frac{a^2 [(1 - \cos 2\theta) + \frac{1 - \cos 2\theta}{2}]}{r^4} \\ &= \frac{a^2 [(1 - \cos 2\theta) + \frac{1 - \cos 2\theta}{2}]}{r^4} \\ &= \frac{a^2 [2 - 2 \cos 2\theta]}{r^4} \\ &= \frac{2a^2}{r^4} \times \frac{r^2}{a^3} \times \frac{a}{r} \\ \therefore \frac{1}{\rho^2} &= \frac{2a}{r^3} \\ \rho^2 &= \frac{r^3}{2a} \end{aligned}$$

Which is the required p-r equation of the curve $r=a(1-\cos 2\theta)$

(2) Find the p-r equation of the

$$r \sin 2\theta + a = 0$$

$$r \sin 2\theta = -a$$

$$r = -a \cos 2\theta$$

$$\frac{dr}{d\theta} = a[\sec^2\theta - \cos^2\theta]$$

$$= a \cos 2\theta$$

we know

$$\frac{1}{\rho^2} = \frac{1}{r^2} + \frac{1}{r^2} \left(\frac{dr}{d\theta} \right)^2$$

$$= \frac{1}{r^2} + \frac{1}{r^2} a^2 \cos^2 2\theta$$

$$\frac{1}{\rho^2} = \frac{1}{r^2} + \frac{1}{r^2} a^2 \cos^2 2\theta$$

$$= \frac{1}{r^2} + \frac{1}{r^2} \left[\frac{r^2 - a^2}{a} \right]$$

$$= \frac{1}{r^2} + \frac{1}{r^2} \cdot \frac{r^2}{a} \cdot \frac{1}{r} = \frac{1}{a}$$

$$\frac{1}{\rho^2} = \frac{1}{a}$$

$$\therefore \rho^2 = a^2$$

$$(2) \frac{l}{r} = 1 + e \cos 2\theta \quad (1)$$

$$\therefore r = \frac{l}{1 + e \cos 2\theta}$$

$$= l(1 + e \cos 2\theta)^{-1}$$

$$\frac{dy}{dx} = \frac{y}{x} \left[-1 + \frac{e \sin 2\theta}{1 + e \cos 2\theta} \right]$$

$$= \frac{y}{x} \left[\frac{e \sin 2\theta}{1 + e \cos 2\theta} \right]$$

$$= \frac{l e \sin 2\theta}{(1 + e \cos 2\theta)^2}$$

$$\left(\frac{dy}{dx} \right)^2 = \frac{l^2 e^2 \sin^2 2\theta}{(1 + e \cos 2\theta)^2}$$

$$= \frac{l^2 e^2 \sin^2 2\theta}{l^2 / r^2}, \text{ using } \dots (1)$$

$$= e^2 r^2 \sin^2 2\theta$$

we know

$$\begin{aligned} \frac{1}{\rho^2} &= \frac{1}{r^2} + \frac{1}{r^2} \left(\frac{dr}{d\theta} \right)^2 \\ &= \frac{1}{r^2} + \frac{1}{r^2} e^{2s} r^2 i^2 \sin^2 \theta \\ &= \frac{1}{r^2} + \frac{1}{r^2} e^{2s} i^2 \sin^2 \theta \\ &= \frac{1}{r^2} (1 + e^{2s} \sin^2 \theta) \end{aligned}$$

But $\frac{l}{r} = 1 + e \cos \theta$

$$\frac{l}{r} - 1 = e \cos \theta$$

$$\begin{aligned} \left(\frac{l}{r} - 1 \right)^2 &= e^2 \cos^2 \theta \\ &= e^2 (1 - \sin^2 \theta) \end{aligned}$$

$$\left(\frac{l}{r} - 1 \right)^2 = e^2 - e^2 \sin^2 \theta$$

$$e^2 \sin^2 \theta = e^2 - \left(\frac{l}{r} - 1 \right)^2$$

using in (2)

$$\frac{1}{\rho^2} = \frac{1}{r^2} \left[1 + e^2 - \left(\frac{l}{r} - 1 \right)^2 \right]$$

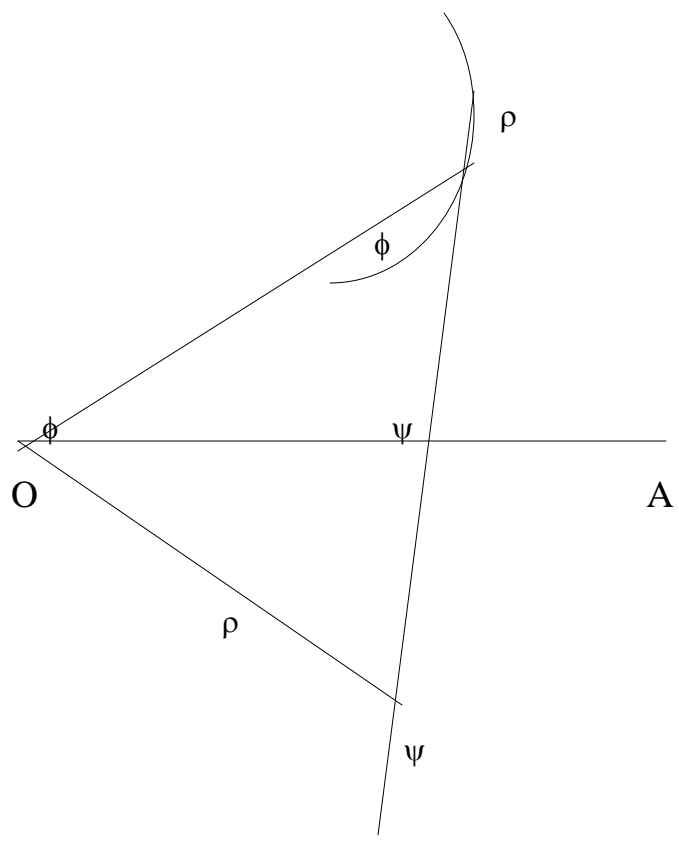
$$= \frac{1}{r^2} \left[1 + e^2 - \left(\frac{l^2}{r^2} - \frac{2l}{r} + 1 \right) \right]$$

$$= \frac{1}{r^2} \left[1 + e^2 - \frac{l^2}{r^2} + \frac{2l}{r} - 1 \right]$$

$$\frac{1}{\rho^2} = \frac{1}{r^2} \left[\frac{2l}{r} + e^2 - \frac{l^2}{r^2} \right]$$

(3) Prove that for the curve $r=f(v)$,

$$\rho = r \frac{d}{d\rho}$$



Let OA be the initial line. Let /o be the pole and be any point on it.
 let the tangent at make an angle with the initial line on let AOP=0. Draw
 ON the tangent at . let ON=

$$\text{let } O = r$$

Also we know

$$\text{si y} = r \frac{d\theta}{d} \frac{d}{s} \frac{r}{d} \frac{r}{s}$$

$$\text{t a y} = r \frac{d\theta}{d}$$

$$p = r \sin 2$$

$$\frac{d}{d} = s \frac{r}{r} + r \frac{d}{d} \frac{\theta}{r}$$

$$= s \theta + nr \theta \frac{d\theta}{d} \frac{d}{s} \frac{d}{d}$$

$$= r \frac{d\theta}{d} + r \frac{d\theta}{d}$$

$$= r \frac{d}{ds} \left(\frac{1}{r} \right)$$

$$\frac{d\rho}{d\psi} = r \frac{d\psi}{ds}$$

$$\frac{1}{r} \frac{ds}{d\psi} = \frac{d}{d\rho}$$

$$\frac{ds}{d\psi} = r \frac{d\rho}{d\psi}$$

$$\therefore l = r \frac{d\rho}{d\psi}$$

Note : For some curve $r = f(\theta)$, sometimes it is not easy to find radius of

curvature, in this case, we should use $l = r \frac{d\rho}{d\psi}$

(4) Show that in the cardio

Proof = $r = a(1 + \cos 2\theta)$

$$\frac{dr}{d\theta} = a(-2 \sin 2\theta)$$

$$\frac{dr}{d\theta} = -2a \sin 2\theta$$

$$\left(\frac{dr}{d\theta} \right)^2 = 4a^2 \sin^2 2\theta$$

we know

$$\frac{1}{\rho^2} = \frac{1}{r^2} + \frac{1}{r^3} \left(\frac{dr}{d\theta} \right)^2$$

$$= \frac{r^2 + \frac{dr^2}{d\theta^2}}{r^4}$$

$$= \frac{r^2 + (4a^2 \sin^2 2\theta)}{r^4}$$

$$= \frac{a^2 [(1 + \cos 2\theta)^2 + 4 \sin^2 2\theta]}{r^4}$$

$$= \frac{a^2 (2 + 2 \cos 2\theta)}{r^4}$$

$$= \frac{a^2 2 (1 + \cos \theta)}{r^4}$$

$$= \frac{2a^2}{r^4} \times \frac{2}{a} = \frac{2a}{r^3}$$

$$\frac{1}{p^2} = \frac{2a}{r^3}$$

$$p^2 = \frac{r^3}{2a} \dots (1)$$

$$2p = \frac{d\rho}{dr} = \frac{1}{r^2} \cdot \frac{3r^2}{2a}$$

$$\frac{d\rho}{dr} = \frac{1}{r^2} \cdot \frac{3r^2}{2a}$$

$$= \frac{3r^2}{4a\rho}$$

$$\therefore \frac{d}{d\rho} = \frac{r^4}{\rho^2}$$

$$r \frac{d}{d\rho} = \frac{r^4}{\rho^2}$$

$$\therefore \rho = \frac{4a\rho}{3r}$$

$$\rho^2 = \frac{1}{9r^2} \rho^2$$

$$\rho^2 = \frac{1}{9r^2} \times \frac{r^3}{2a}, \text{ using (1)}$$

$$\rho^2 = \frac{2a}{9}$$

$$\frac{\rho^2}{r} = \frac{8a}{9} = a \text{ constant}$$

15.6. Let us sum up

So far we have studied the method of finding the radius of curvature in polar form, involutes, evolutes and finding the p-r equation of a curve.

15.7. Check your progress

(1) Show that in the cardioid $r=a(1+\cos 2\theta)$, $\frac{l^2}{r}$ is a constant

(2) Show that the evolutes of the cycloid $x=a(2-x\sin 2)$, $y=a(1-\cos)$ is another cycloid

15.8. Lesson End Activities

(1) For the polar equation of the parabola, $\frac{2a}{r} = 1 - \cos \theta$ show that $p^2 = ar$

(2) Prove that the p-r equation of the curve $r^2 = a^2 \sin^2 \theta$ is $p = \frac{r^3}{a^2}$ Hence show

that $l = \frac{a^2}{3r}$

(3) Prove that the radius of curvature of the curve $r \cos^2(\theta/2) = a$ is $2a \sec^3(\theta/2)$

(4) Show that the radius curvature of the curve $r^2 = a^2 \cos 2\theta$ is $\frac{a^2}{3r}$

15.9. Points for discussion

(1) Prove that the p-r equation of the curve $r = a \cot \alpha$ is

(2) Find the p-r equation of the curve $x^2 + y^2 = ax$ and deduce its radius of

curvature $\left(\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} \right) \frac{1}{a} = \frac{r^2}{b}$

(7) For the curve $r = a \sin \theta$, show that the p-r equation is $ap = r^2$

15.10. References

A book of Calculus by T.K. Manickavasagam Pillai

Lesson-16

TOTAL DIFFERENTIATION

Contents

- 16.0 Aims and Objectives
- 16.1. Definition
- 16.2 Example
- 16.3 Euler's Theorem on homogeneous functions
- 16.4. Examples (1) verify Euler's theorem for
- 16.5. Let us sum up
- 16.6. Check your progress
- 16.7. Lesson End Activities
- 16.8. Points for discussion
- 16.9 References

16.0 Aims and Objectives

Our aim is to learn the total differentiation of a function with respect to a veritable using partial derivatives.

16.1. Definition(1) : Let u be a function of namely where x and y are functions of some other variable 't'.

Then $\frac{d}{dt} u = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$ is called total differentiation.

Also $d u = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$ is called total differential of u .

Definition(2) : Let u be a function of x and y where x and y are functions of t

then $\frac{d}{dt} u = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt}$

we replace t by x

$$\therefore \frac{d}{dx} u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \frac{dy}{dx}$$

Definition 3

Implicit functions

Let the relation between x and y be with form $f(x,y)=c$ where c is a constant then the total differential coefficient with respect to x is zero.

$$\therefore 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

16.2 Example

Find $\frac{d}{dt}$ if $u = xyz$ where $x=e^{-t}$; $y=e^{-t}\sin^2 t$; $z=\sin t$

Solution : $u=xy^2z$

$$\frac{\partial u}{\partial x} = y^2z$$

$$\frac{\partial u}{\partial y} = 2xy z$$

$$\frac{\partial u}{\partial z} = xy^2$$

$$x=e^{-t}$$

$$y=e^{-t}\sin^2 t$$

$$\frac{dx}{dt} = -e^{-t}$$

$$\frac{dy}{dt} = e^{-t} \times 2 \sin t \cos t - e^{-t} \sin^2 t$$

$$= e^{-t} [2 \sin t \cos t - \sin^2 t]$$

$$z=\sin t$$

$$\frac{dz}{dt} = \cos t$$

$$\frac{d}{dt} \frac{d}{dt} \frac{\partial u}{\partial x} + \frac{d}{dt} \frac{\partial u}{\partial y} \frac{dy}{dt} + \frac{d}{dt} \frac{\partial u}{\partial z} \frac{dz}{dt}$$

$$= y^2z (-e^{-t}) + (2xy z - e^{-t} \sin^2 t) (2 \sin t \cos t - \sin^2 t) + xy^2 \cos t$$

$$= e^{-t} \sin^2 t \cos t - e^{-t} \sin^2 t (2 \sin t \cos t - \sin^2 t) + e^{-t} \sin^2 t \cos t$$

$$= e^{-2t} \sin^2 t \cos t - (2 \sin^3 t \cos t - \sin^4 t) + e^{-t} \sin^2 t \cos t$$

$$= e^{-2t} [\sin^3 t \cos t - 2 \sin^3 t \cos t + \sin^4 t + \sin^2 t \cos t]$$

$$= e^{-2t} [3 \sin^2 t - 3 \cos^2 t]$$

$$= e^{-2t} \sin^2 t \left[\frac{\cos^2 t}{\sin^2 t} - 3 \right]$$

$$\frac{d}{dt} u = e^{-2t} \sin^2 t [3 \cos^2 t - 3]$$

(2) Find $\frac{d}{dt} u$ if $u = x^2 + y^2 + a^2$; $x^3 + y^3 = a^3$

solution

$$\frac{d}{dt} u = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} + \dots$$

$$u = x^2 + y^2 + a^2$$

$$\frac{\partial u}{\partial x} = 2x \quad \frac{\partial u}{\partial y} = 2y$$

$$x^3 + y^3 = a^3$$

diff. both sides w.r.t.x

$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$

$$3y^2 \frac{dy}{dx} = -3x^2$$

$$\frac{dy}{dx} = -\frac{x^2}{y^2}$$

using in(1)

$$\frac{d}{dt} u = 2x \frac{dx}{dt} + 2y \left(\frac{dx}{dt} \frac{x^2}{y^2} \right)$$

$$= 2x - \frac{2x^2}{y}$$

$$= \frac{1}{y} (x^2 - y^2)$$

$$= \frac{2x}{y} (y - x)$$

(3) Find $\frac{d}{dt} u$ if x and y are related as

$$y^2 = 4ax$$

$$f(x,y) = y^2 - 4ax$$

$$\frac{\partial f}{\partial x} = -4a$$

$$\frac{\partial f}{\partial y} = 2y$$

$$\frac{d}{dx} \left(\frac{y}{f} \right) = \frac{-x \cdot 4a}{y^2 \cdot y}$$

$$= \frac{2a}{y}$$

(4) Find $\frac{d}{dx}$ if $ax^2 + 2bxy + by^2 + 2gx + 2fy + c = 0$

Solution

$$f = ax^2 + 2bxy + by^2 + 2gx + 2fy + c$$

$$\frac{\partial f}{\partial x} = 2ax + 2by + 2g$$

$$\frac{\partial f}{\partial y} = 2bx + 2by + 2f$$

$$\frac{d}{dx} \left(\frac{y}{f} \right) = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$= \frac{2ax + 2by + 2g}{2bx + 2by + 2f}$$

$$= \frac{(a + h)y + g}{2h + 2b + 2f}$$

16.3 Euler's Theorem on homogeneous functions

Definition 1: Let u be a function of x and y . If $u(tx, ty) = t^n u(x, y)$ then $u(x, y)$ is called a homogeneous function of degree n in x and y .

Eg: $u(x, y) = ax^2 + 2hxy + by^2$

$$\begin{aligned} u(tx, ty) &= at^2x^2 + 2ht^2xy + bt^2y^2 \\ &= t^2(ax^2 + 2hxy + by^2) \end{aligned}$$

$$=t^2u(x,y)$$

u (x,y) is a homogeneous function of 2nd degree in x and y

Eule's theorem on homogenous function

If u be a homogeneous function of nth degree in x and y then

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$$

This is called Endle's theorem on homogeneous functions.

Proof:

$$\text{Let } u(x,y) = a_0x^2 + a_1x^{n-1}y + a_2x^{n-2}y^2 + a_3x^{n-3}y^3 + \dots + any^n$$

$$= x^n \left[a_0 + a_1 \frac{y}{x} + a_2 \frac{y^2}{x^2} + a_3 \frac{y^3}{x^3} + \dots + a_n \frac{y^n}{x^n} \right]$$

$$= x^n \left[a_0 + a_1 \frac{y}{x} + a_2 \left(\frac{y^2}{x^2} \right) + a_3 \left(\frac{y^3}{x^3} \right) + \dots + a_n \left(\frac{y^n}{x^n} \right) \right]$$

$$u = x^n f(y/x)$$

$$\frac{\partial u}{\partial x} = x^{n-1} f(y/x) + f(y/x) x^n \left(-\frac{y}{x^2} \right)$$

$$= y^{n-2} x \left(\frac{y}{x} \right)^{n-1} f(y/x) - x^{n-1} y f'(y/x)$$

$$\therefore x \frac{\partial u}{\partial x} = y^{n-1} f(y/x) - x^{n-1} y f'(y/x)$$

$$= x^{n-1} f(y/x)$$

$$\therefore y \frac{\partial u}{\partial y} = x^{n-1} y f'(y/x)$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x^{n-1} f(y/x) + x^{n-1} y f'(y/x)$$

$$= n x^{n-1} f(y/x)$$

$$= nu$$

16.4. Examples (1) verify Euler's theorem for

$$u = x^3 + y^3 + z^3 + 3xy^2$$

Solution

$$\frac{\partial u}{\partial x} = 3x^2 + 3y^2$$

$$\frac{\partial u}{\partial y} = 3y^2 + 3z$$

$$\frac{\partial u}{\partial z} = 3z^2 + 3y$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 3x^3 + 3y^3 + 3z^3 + 3xy + 3xz + 3yz$$

$$= 3x^3 + 3y^3 + 3z^3 + 9xyz$$

$$= 3(x^3 + y^3 + z^3 + 3xyz)$$

$$= 3u$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 3u$$

\therefore Euler's theorem is verified

(2) If $u = \sin^{-1} \left(\frac{x^2 + y^2}{x + y} \right)$, show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u$$

Solution

$$\text{given } u = \sin^{-1} \left(\frac{x^2 + y^2}{x + y} \right)$$

$$\therefore \sin u = \frac{x^2 + y^2}{x + y}$$

let $f = \sin x$

$$\therefore f = \frac{x^2 + y^2}{x + y}$$

$$f(tx, ty) = \frac{t^2(x^2 + y^2)}{tx + ty}$$

$$= \frac{t^2(x^2 + y^2)}{x + y}$$

$$f(tx, ty) = t^2 f(x, y)$$

\therefore f is a homogeneous function of 2nd degree in x and y .

\therefore By Euler's

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = f$$

$$x \frac{\partial}{\partial x} (s^{-1}) + y \frac{\partial}{\partial y} (s^{-1}) = -2 s^{-1}$$

$$x c \frac{\partial u}{\partial x} + y c \frac{\partial u}{\partial y} = -2 s^{-1}$$

$$c \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] = -2 s^{-1}$$

$$c \left[x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right] = -2 s^{-1}$$

16.5. Let us sum up

Solve have learnt on finding the total differentiation and Euler's Theorem on homogeneous function.

16.6. Check your progress

(1) Find $\frac{d}{dt}$ if $u=xyz$; where $x=t, y=t^2, z=t^3$

(2) Find $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}$ if $u=x+y+z$

16.7. Lesson End Activities

Exercise

Verify Euler's theorem on homogeneous functions

1) (a) $\frac{x^{1/4} + y^{1/4}}{x^{1/3} + y^{1/3}}$

(b) $u=ax^2+2hxy+by^2$

(c) $u=x^3-3x^2y+3x^2y+y^3$

(d) $u = \frac{x^2(x-y)^3}{(x^2+y)^3}$

2) If $u = s^{-1} \left(\frac{x+y}{\sqrt{x} + \sqrt{y}} \right)$, prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} u$$

3) If $u = t^{-1} \left(\frac{x^2 + y^2}{x + y} \right)$, show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} u \quad \text{in}$$

4) If u be a homogeneous function of degree n in x and y, prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = n u \quad \text{hence show that} \quad x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

5) If $u = \tan^{-1} \left(\frac{x^3 + y^3}{x - y} \right)$, prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} u \quad \text{in}$$

6) Find $\frac{d}{dt} u$ if $u = x^3 y^4 z^2$ where $x=t^2, y=t^3, z=t^4$

7) Find $\frac{d}{dt} u$ if $u = \sin^2 x$ where $x = \log t, y = e^t$

8) Find $\frac{d}{dt} u$ if $u = x^3 y^2$ where $x^2 - xy + y^2 = a^2$

16.8. Points for discussion

9) Find $\frac{d}{dt}$ if (a) $xy = e^2$

(b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(c) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(d) $x^2 + y^2 = a^2$

(f) $(x+a)^2 + (y+b)^2 = r^2$

(g) $x^{2/3} + y^{2/3}$

(h) $(y-k)^2 = 4a(x-h)$

(i) $ax^2 + 2hxy + by^2 = 0$

(j) $(x-h)(y-k) = c^2$

16.9 References

A book of Calculus by T.K. Manickavasagam Pillai

Lesson-17

INTEGRATION

Contents

- 17.0. Aims and Objectives
- 17.1. Integrals
- 17.2. Examples
- 17.3. Let us sum up
- 17.4. Check your progress
- 17.5. Lesson End Activities
- 17.6 Points for discussion
- 17.7. References

17.0 Aims and Objectives

In this method different methods of integration are to be learnt

Methods of Integration

17.1 Integrals of the form $\int \frac{f'(x) dx}{f(x)}$ Method proof $f(x) = t$

17.2. Examples $\int \frac{\sin x dx}{\cos x}$

put $\cos x = t$

$\therefore -\sin x dx = dt$

$\sin x dx = -dt$

$$I = \int \frac{\sin x dx}{\cos x}$$

$$= \int \frac{-dt}{t}$$

$$= -\log t + c$$

$$= -\log \cos x + c$$

$$= +\log(\cos x)^{-1} + c$$

$$= \log \frac{1}{\cos x} + c$$

$$= \log \sin x + c$$

(2) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

put $e^x + e^{-x} = t$

$$(e^x - e^{-x})dx = dt$$

$$I = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$\int \frac{dt}{t} = \log t + c$$

$$= \log(e^x - e^{-x})$$

$$(3) \int \frac{x dx}{\sqrt{1+x^2}} \text{ put } 1+x^2 = t$$

$$2x dx = dt$$

$$x dx = \frac{dt}{2}$$

$$\therefore I = \int \frac{x dx}{\sqrt{1+x^2}}$$

$$= \int \frac{dt/2}{\sqrt{t}}$$

$$= \frac{1}{2} \int t^{-1/2} dt = \frac{t^{1/2}}{1/2} + c$$

$$= \sqrt{t} + c$$

$$= \sqrt{1+x^2} + c$$

17.2.1. Integrals of the form

$$\int \frac{(px + q) dx}{a^2x^2 + b} = \frac{M}{p} \frac{dx}{a^2x^2 + b} + \frac{N}{q} \frac{dx}{x} + c$$

Important formulas

$$(1) \int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + c$$

$$(2) \int \frac{dx}{x^2 + a^2} = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + c$$

$$(3) \int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{x+a}{x-a} \right| + c$$

$$(1) \text{ Example } \int \frac{x dx}{x^2 + x + 1}$$

$$\text{let } x = \frac{d}{dx} (x^2 + x) + B$$

$$x = A(2x+1) + B \dots (1)$$

compose the coefficient of x

$$1 = 2A \quad \therefore A = 1/2$$

put x = out ... (1)

$$0 = A + B \quad \therefore B = -1/2$$

$$\therefore x = 1/2 (2x+1)^{-1/2}$$

$$\begin{aligned} \text{let } I &= \int \frac{x \, dx}{x^2 + x + 1} \\ &= \int \frac{\frac{1}{2}(2x+1) - \frac{1}{2}}{x^2 + x + 1} dx \\ &= \frac{1}{2} \int \frac{2x+1}{x^2+x+1} dx - \frac{1}{2} \int \frac{dx}{x^2+x+1} \\ &= \frac{1}{2} \int \frac{2x+1}{x^2+x+\frac{1}{4} + \frac{3}{4}} dx - \frac{1}{2} \int \frac{dx}{(x+\frac{1}{2})^2 + \frac{3}{4}} \\ &= \frac{1}{2} \int \frac{2x+1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx - \frac{1}{2} \int \frac{dx}{(x+\frac{1}{2})^2 + \frac{3}{4}} \\ &= \frac{1}{2} \int \frac{2x+1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx - \frac{1}{2} \int \frac{dx}{(x+\frac{1}{2})^2 + \frac{3}{4}} \\ &= \frac{1}{2} \int \frac{2x+1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx - \frac{1}{2} \int \frac{dx}{(x+\frac{1}{2})^2 + \frac{3}{4}} \\ &= \frac{1}{2} \int \frac{2x+1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx - \frac{1}{2} \int \frac{dx}{(x+\frac{1}{2})^2 + \frac{3}{4}} \\ &= \frac{1}{2} \int \frac{2x+1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx - \frac{1}{2} \int \frac{dx}{(x+\frac{1}{2})^2 + \frac{3}{4}} \\ &= \frac{1}{2} \int \frac{2x+1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx - \frac{1}{2} \int \frac{dx}{(x+\frac{1}{2})^2 + \frac{3}{4}} \\ &= \frac{1}{2} \int \frac{2x+1}{(x+\frac{1}{2})^2 + \frac{3}{4}} dx - \frac{1}{2} \int \frac{dx}{(x+\frac{1}{2})^2 + \frac{3}{4}} \end{aligned}$$

$$(2) \int \frac{3x+2}{3x^2-4x+2} dx$$

$$I = \int \frac{3x+2}{3x^2-4x+2} dx \dots (1)$$

$$\text{Let } 3x+2 = A \frac{d}{dx} (3x^2-4x+2) + B$$

$$3x+2 = A(6x-4) + B$$

put x = 0

$$2 = -4A + B \dots (2)$$

put $x=1$

$$5 = 2A + B \dots (3)$$

solve (2) and (3)

$$A = 1/2; B = 4$$

$$\therefore 3x + 2 = 1/2(6x - 4) + 4$$

using in (1)

$$\begin{aligned} I &= \int \frac{1}{2} \frac{(6x - 4) + 8}{3x^2 - 4x + 2} dx \\ &= \frac{1}{2} \int \frac{6x - 4}{3x^2 - 4x + 2} + \frac{4}{3x^2 - 4x + 2} dx \\ &= \frac{1}{2} \int \frac{d}{dx} \left(\frac{2x - 2}{3x^2 - 4x + 2} \right) + \int \frac{4}{3x^2 - 4x + 2} dx \\ &= \frac{1}{2} \ln \left| \frac{2x - 2}{3x^2 - 4x + 2} \right| + \int \frac{4}{3x^2 - 4x + 2} dx \\ &= \frac{1}{2} \ln \left| \frac{2x - 2}{3x^2 - 4x + 2} \right| + \int \frac{4}{\left(x - \frac{2}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2} dx \\ &= \frac{1}{2} \ln \left| \frac{2x - 2}{3x^2 - 4x + 2} \right| + \frac{4}{\frac{\sqrt{2}}{3}} \arctan \left(\frac{x - \frac{2}{3}}{\frac{\sqrt{2}}{3}} \right) \\ &= \frac{1}{2} \ln \left| \frac{2x - 2}{3x^2 - 4x + 2} \right| + \frac{4\sqrt{2}}{3} \arctan \left(\frac{2x - 2}{\sqrt{2}} \right) \\ &= \frac{1}{2} \ln \left| \frac{2x - 2}{3x^2 - 4x + 2} \right| + \frac{4\sqrt{2}}{3} \arctan \left(\frac{x - 1}{\sqrt{2}} \right) \end{aligned}$$

$$(3) \int \frac{x+1}{1-x-x^2} dx$$

$$\text{let } x+1 = A \frac{d}{dx}(-x^2 - x) + B$$

$$x+1 = A(-1-2x) + B$$

$$\text{put } x=0, 1 = -A+B$$

$$\text{put } x = -1/2 : 1/2 = B$$

$$\therefore A = -1/2$$

$$\therefore x+1 = -1/2 (-1-2x) + 1/2$$

$$I = \int \frac{x+1}{1-x-x^2} dx$$

$$= \int \frac{-1/2(-1-x) + 1/2}{1-x-x^2} dx$$

$$= \frac{1}{2} \int \frac{1-x}{1-x-x^2} dx + \frac{1}{2} \int \frac{dx}{1-x-x^2}$$

$$= \frac{1}{2} \int \frac{1-x}{1-(x^2+x)} dx + \frac{1}{2} \int \frac{dx}{1-(x^2+x)}$$

$$= \frac{1}{2} \int \frac{1-x}{1-\left(x^2 + \frac{1}{2}x + \frac{1}{4}\right) - \frac{1}{4}} dx$$

$$= \frac{1}{2} \int \frac{1-x}{1-\left(x + \frac{1}{2}\right)^2 - \frac{1}{4}} dx$$

$$= \frac{1}{2} \int \frac{1-x}{3/4 - \left(x + 1/2\right)^2} dx$$

$$= \frac{1}{2} \int \frac{1-x}{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(x + 1/2\right)^2} dx$$

$$= \frac{1}{2} \int \frac{1-x}{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(x + 1/2\right)^2} dx = \frac{1}{2} \int \frac{1-x}{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(x + 1/2\right)^2} dx$$

$$= \frac{1}{2} \int \frac{1-x}{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(x + 1/2\right)^2} dx = \frac{1}{2} \int \frac{1-x}{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(x + 1/2\right)^2} dx$$

17.2.3. Integrals of the form

(a) $\int \sqrt{\frac{x-\alpha}{\beta-x}}$ in where $\beta > \alpha$

Example (1) $\int \sqrt{\frac{x-2}{5-x}} dx$

let $x = \alpha \cos^2\theta + \beta \sin^2\theta$ (Method)

Here $\alpha = 2, \beta = 5$

$$x = 2 \cos^2\theta + 5 \sin^2\theta$$

$$dx = [2(2\cos\theta \sin\theta) + 5(2\sin\theta \cos\theta)] d\theta$$

$$= [-4\sin\theta \cos\theta + 10\sin\theta \cos\theta] d\theta$$

$$= 6\sin\theta \cos\theta d\theta$$

$$x-2 = 2\cos^2\theta + 5\sin^2\theta - 2$$

$$= 2(\cos^2\theta - 1) + 5\sin^2\theta$$

$$= 2(-\sin^2\theta) + 5\sin^2\theta$$

$$= -2\sin^2\theta + 5\sin^2\theta$$

$$\boxed{x-2 = 3\sin^2\theta}$$

$$5-x = 5 - (2\cos^2\theta + 5\sin^2\theta)$$

$$= 5 - 2\cos^2\theta - 5\sin^2\theta$$

$$= 5(1 - \sin^2\theta) - 2\cos^2\theta$$

$$= 5\cos^2\theta - 2\cos^2\theta$$

$$\boxed{5-x = 3\cos^2\theta}$$

$$I = \int \sqrt{\frac{x-2}{5-x}} dx$$

$$= \int \sqrt{\frac{3\sin^2\theta}{3\cos^2\theta}} \cdot 6\sin\theta \cos\theta d\theta$$

$$= \int \frac{\sin\theta}{\cos\theta} \cdot 6\sin\theta \cos\theta d\theta$$

$$= \int 6\sin^2\theta d\theta$$

$$= 3 \int 2\sin^2\theta d\theta$$

$$\begin{aligned}
 &= 3 \int (1 - \cos 2\theta) d\theta \quad \{ \therefore 1 - \cos 2\theta = 2\sin^2\theta \} \\
 &= 3 \left[\int d\theta - \int \cos 2\theta d\theta \right] \\
 &= 3 \left[\theta - \frac{\sin 2\theta}{2} \right] + c \\
 &= 3 \left[\theta - \frac{2 \sin \theta \cos \theta}{2} \right] + c \\
 &= 3[2 - \sin 2\theta] + c
 \end{aligned}$$

But $x-2 = 3\sin^2\theta$

$$\frac{x-2}{3} = \sin^2\theta$$

$$\sin\theta = \sqrt{\frac{x-2}{3}}$$

$$\theta = \sin^{-1} \sqrt{\frac{x-2}{3}}$$

Also $\cos 2\theta = \sqrt{\frac{5-x}{3}}$

using in (1)

$$\begin{aligned}
 I &= 3 \left[\sin^{-1} \sqrt{\frac{x-2}{3}} - \frac{\sqrt{\frac{x-2}{3}} \sqrt{\frac{5-x}{3}}}{\frac{1}{3}} \right] + c \\
 &= 3 \left[\sin^{-1} \sqrt{\frac{x-2}{3}} - \frac{1}{3} \sqrt{(x-2)(5-x)} \right] + c
 \end{aligned}$$

(b) Integrals of the form $\int \sqrt{(x-a)(b-x)} dx$

Method As in (a)

problem $\int \sqrt{(x-2)(5-x)} dx$

$$I = \int \sqrt{(x-2)(5-x)} dx$$

$$= \int \sqrt{3 \sin^2\theta \cdot 3 \cos^2\theta} d\theta$$

$$= 1 \int \sin^2\theta \cos^2\theta d\theta$$

$$= \frac{1}{4} \int 4 \sin^2\theta \cos^2\theta d\theta$$

$$\begin{aligned}
 &= \frac{9}{2} \int (\sin \theta)^2 d\theta \\
 &= \frac{9}{2} \int (\sin \theta)^2 d\theta \\
 &= \frac{9}{2} \int \sin^2 \theta d\theta \\
 &= \frac{9}{2} \int \frac{1 - \cos 2\theta}{2} d\theta \quad \because \sin^2 \theta = \frac{1 - \cos 2\theta}{2} \\
 & \quad \text{Hence } \alpha = 2\theta \\
 &= \frac{9}{4} \int (1 - \cos 2\theta) d\theta \\
 &= \frac{9}{4} \left[\int d\theta - \int \cos 2\theta \right] \\
 &= \frac{9}{4} \left[\theta - \frac{\sin 2\theta}{2} \right] + c \\
 &= \frac{9}{4} \left[\theta - \frac{\sin 2\theta}{2} \right] + c \\
 &= \frac{9}{4} \left[\theta - \frac{2 \sin \theta \cos \theta}{2} \right] + c \\
 &= \frac{9}{4} \left[\theta - \sin \theta \cos \theta \right] + c \dots
 \end{aligned}$$

But $\sin 2\theta = \sqrt{\frac{x-2}{3}}$ and $\cos 2\theta = \sqrt{\frac{5-x}{3}}$

$$\cos 2\theta = 2\cos^2 \theta - 1$$

$$= 2 \left[\frac{5-x}{3} \right]$$

$$= \frac{1}{3} [10 - 2x]$$

$$= \frac{1}{3} [7 - 2x]$$

using in (1)

$$I = \frac{9}{4} \left[\theta - \frac{2 \sqrt{\frac{x-2}{3}} \sqrt{\frac{5-x}{3}}}{2} \right] + c$$

$$I = \frac{9}{4} \left[s^{-2} \sqrt{\frac{x-1}{3}} - \frac{1}{9} x \sqrt{(3-x)} \right] x(-7-2ct)$$

(c) Integrals of the form

$$\int \frac{dx}{\sqrt{(x-a)^2 + b}}$$

Method as in (a)

problem

$$I = \int \frac{dx}{\sqrt{(x-2)^2 + 5}}$$

$$x-2 = 3 \sin^2 v$$

$$5-x = 3 \cos^2 v$$

$$dx = 6 \sin v \cos v dv$$

$$I = \int \frac{dx}{\sqrt{(x-2)^2 + 5}}$$

$$= \int \frac{6 \sin v \cos v dv}{\sqrt{(3 \sin^2 v) + (3 \cos^2 v)}}$$

$$= \int \frac{6 \sin v \cos v dv}{3 \sin^2 v + 3 \cos^2 v}$$

$$= 2 \int dv$$

$$= 2v + c$$

$$= 2 \sin^{-1} \left(\sqrt{\frac{x-2}{3}} \right) + c \quad \left[\begin{array}{l} \therefore 3 \sin^2 v = x-2 \\ \sin v = \sqrt{\frac{x-2}{3}} \end{array} \right]$$

$$\sin v = \sqrt{\frac{x-2}{3}}$$

17.2.4. Integrals of the form

$$\int \frac{1}{(a \cos x + b \sin x + c)} dx$$

Method put $\sin x = \frac{2t}{1+t^2}$

$$\cos x = \frac{1-t^2}{1+t^2}$$

where $t = \tan(x/2)$

Problem (1) $\int \frac{dx}{1 + \sin x}$

$$\text{put } \sin x = \frac{2 \tan(x/2)}{1 + \tan^2(x/2)}$$

$$\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$$

$$t = \tan(x/2)$$

$$dt = \sec^2(x/2) \cdot \frac{1}{2} dx$$

$$dx = \frac{2 dt}{1 + t^2}$$

$$= \frac{2 dt}{1 + t^2}$$

$$dx = \frac{2 dt}{1 + t^2}$$

Let $I = \int \frac{dx}{1 + \sin x}$

$$= \int \frac{\frac{2 dt}{1 + t^2}}{1 + \frac{2t}{1 + t^2}}$$

$$= \int \frac{dx}{2t^2 + 2t + 1}$$

$$= \int \frac{dt}{2t + 1}$$

$$= \int \frac{dt}{t + 2}$$

$$= \log(t + 1) + c$$

$$= \log[\tan(x/2) + 1] + c$$

(2) $\int \frac{dx}{5 + 4 \cos x - 3 \sin x}$

$$I = \int \frac{dx}{5 + 4 \cos x - 3 \sin x}$$

$$\begin{aligned}
 &= \int \frac{\frac{2d}{1+t^2}}{5 + \frac{4x(-1^2)}{1+t^2} + \frac{x^3}{1+t^2}} \\
 &= 2 \int \frac{d}{5 + t^2 + t^4 - (f1 -) t} \\
 &= 2 \int \frac{d}{5 + 5t^2 + 4t^2 + 4f - t} \\
 &= 2 \int \frac{d}{t^2 - f + 9} \\
 &= 2 \int \frac{d}{(t-3)^2} \\
 &= 2 \left[\frac{(t-3)^{-1}}{-1} \right] + c \\
 &= 2 \left[\frac{1}{t-3} \right] + c \\
 &= 2 \left[\frac{1}{\tan^{-1}(\frac{1}{2}) + 3} \right] + c
 \end{aligned}$$

17.3. Let us sum up

We have studied different types of evaluation of integrals.

17.4. Check your progress

$$\begin{aligned}
 (1) \int \frac{(e^x + e^{-x})}{e^x + e^{-x}} dx & \qquad (2) \int \frac{dx}{x^2 - x + 1} \\
 (3) \int \frac{dx}{2x^2 - x + 1} & \qquad (4) \int \frac{dx}{1 - x^2 + x^2}
 \end{aligned}$$

17.5. Lesson End Activities

Integrate the following

$$\begin{aligned}
 (1) \int \frac{e^x dx}{e^{2x} + 1} & \qquad (\tan^{-1}(e^x) + c) \\
 (2) \int \frac{\sin(\log x)}{x} dx & \qquad (-\cos(\log x) + c) \\
 (3) \int \frac{dx}{x(1 + x^2)} & \qquad \left(-\frac{1}{(n-1)(\log^{-1})} \right)
 \end{aligned}$$

$$(4) \int \frac{5-4x}{1+2x-x^2} dx \quad A : 2 \ln \left| \frac{2\sqrt{x} + 1}{2\sqrt{x} - 1} \right| + \frac{1}{2} \left[\frac{2\sqrt{x} + 1}{2\sqrt{x} - 1} \right] + c$$

$$(5) \int \frac{3x+1}{2x^2+x^3} dx \quad \left\{ \frac{3}{4} \ln \left| \frac{2\sqrt{x} + 1}{2\sqrt{x} - 1} \right| + \frac{1}{2} \left[\frac{2\sqrt{x} + 1}{2\sqrt{x} - 1} \right] \right\} + c$$

17.6 Points for discussion

$$(6) \int \frac{x+4}{6x^2-7x-2} dx \quad \left\{ -\frac{1}{2} \ln \left| \frac{6x-7}{2\sqrt{x}} \right| + \frac{1}{2} \left[\frac{\sqrt{3x}}{\sqrt{2-x}} \right] \right\} + c$$

$$(7) \int \frac{dx}{1-3 \cos 2x} \quad \left\{ \frac{1}{2\sqrt{3}} \ln \left| \frac{1 + \sqrt{3} \tan(x - \frac{\pi}{4})}{1 - \sqrt{3} \tan(x - \frac{\pi}{4})} \right| \right\} + c$$

$$(8) \int \frac{dx}{1+3 \cos 4x} \quad \left\{ \frac{1}{2\sqrt{3}} \ln \left| \frac{\sqrt{3} \tan(x - \frac{\pi}{4}) + \sqrt{2}}{\sqrt{3} \tan(x - \frac{\pi}{4}) - \sqrt{2}} \right| \right\} + c$$

17.7. References

A book of Integral Calculus by T.K. Manickavasagam Pillai

Lesson – 18

INTEGRATION BY PARTS

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18.0 Aims and Objectives

Our Aim is to evaluate a given integral by parts method

18.1. Integration by parts

We know that $\frac{d}{dx}(uv) = u \frac{d}{dx}v + v \frac{d}{dx}u$

$$\therefore d(uv) = u dv + v du$$

$$\therefore \int d(uv) = \int u dv + \int v du$$

$$\therefore \int u dv = \int v du + \int u dv$$

$$\boxed{\therefore \int u dv = uv - \int v du}$$

18.2. Examples

(1) $\int x \sin 2x dx$

$$u = x; \quad dx = \sin 2x dx$$

$$dv = dx \quad \int d \Rightarrow \int \sin 2x dx$$

$$v = -\frac{\cos 2x}{2}$$

$$I = \int x \sin 2x dx$$

$$= uv - \int v du$$

$$\begin{aligned}
 &= \frac{-xc \cos x}{2} \int \left(\frac{\cos x}{2} \right)^2 dx \\
 &= \frac{-xc \cos x}{2} \cdot \frac{2}{2} \int \cos^2 x dx \\
 &= \frac{-xc \cos x}{2} \cdot \frac{1}{2} \left(\frac{\sin 2x}{2} \right) + c \\
 &= \frac{-xc \cos x}{2} \cdot \frac{1}{2} (\sin 2x) + c
 \end{aligned}$$

(2) $\int x \tan^2 x dx$

$u = x ; dv = \tan^2 x dx$

$du = dx \quad \int dx \Rightarrow \int x \tan^2 x dx$

$v = \int (\sec^2 x - 1) dx$

$v = \tan x - x$

$v = \tan x - x$

$I = \int x \tan^2 x dx$

$= uv - \int v du$

$= x(\tan x - x) - \int (\tan x - x) dx$

$= x(\tan x - x) - \int \tan x dx + \int x dx$

$I = (x \tan x - x^2) - \ln |\sec x| + \frac{x^2}{2} + c$

(3) $\int \log x dx$

$u = \log x ; dv = dx$

$\frac{d}{dx} \log x = \frac{1}{x} \quad \int dx = x$

$dv = \frac{1}{x} dx$

$I = \int \log x dx$

$= uv - \int v du$

$$= x \log x - \int x \, dx$$

$$I = x \log x - \frac{x^2}{2} + c$$

$$(4) \int x^2 \tan^{-1} x \, dx$$

$$u = \tan^{-1} x$$

$$dv = x^2 dx$$

$$du = \frac{1}{1+x^2} dx$$

$$v = \frac{x^3}{3}$$

$$v = \frac{x^3}{3}$$

$$I = \int u \, dv - \int v \, du$$

$$= uv - \int v \, du$$

$$= \frac{x^3}{3} \tan^{-1} x - \int \frac{x^3}{1+x^2} dx$$

$$= \frac{x^3}{3} \tan^{-1} x - \int \frac{x^3}{1+x^2} dx \quad (1)$$

$$\text{But } \frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}$$

$$\frac{x^3+1}{-1}$$

$$\therefore \frac{x^3}{x^3+1} = \frac{x}{x+1}$$

using in (1)

$$I = \frac{x^3}{3} \log x - \frac{1}{3} \int \left(\frac{x}{x^2+1} \right) dx$$

$$= \frac{x^3}{3} \log x - \frac{1}{3} \int \frac{x \, dx}{x^2+1}$$

$$= \frac{x^3}{3} \log x - \frac{1}{3} \left[\frac{x^2}{2} + a \right]^{n+1} + c$$

$$I = \frac{x^3}{3} \log x - \frac{1}{6} \left[\frac{x^2}{2} + a \right]^{n+1} + c$$

$$(5) \int \frac{1 \, dx}{(1+x^2)^2}$$

$$u = \frac{1}{1+x^2}; \quad v = \frac{1}{1+x^2} dx$$

$$d = \frac{1}{x} \quad \int d = \int \frac{dx}{(1+x^2)}$$

$$v = -(1+x)^{-1}$$

$$= \frac{1}{1+x}$$

$$I = \int \frac{1}{(x+1)^2} dx$$

$$= u \cdot \int v \cdot d$$

$$= \frac{1}{x+1} \cdot \int \left(-\frac{1}{x+1} \right) dx$$

$$= \frac{1}{(x+1)^2} \cdot (-1) \cdot \int \frac{1}{x+1} dx$$

To find $\int \frac{1}{x(x+1)} dx$

By partial fraction method

$$\text{Let } \frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \dots (1)$$

$$\frac{1}{x(x+1)} = \frac{A(x+1) + Bx}{x(x+1)}$$

$$1 = A(x+1) + Bx \dots \dots \dots (2)$$

$$\text{Put } x = -1 ; B = -1$$

$$\text{Put } x = 0 ; 1 = A$$

$$\therefore \boxed{A = 1} \quad \boxed{B = -1}$$

using in (3)

$$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$$

$$\int \frac{1}{x(x+1)} dx = \int \left(\frac{1}{x} - \frac{1}{x+1} \right) dx$$

$$= \log x - \log(x+1) + c$$

$$= \log \left(\frac{x}{x+1} \right) + c$$

using in (1)

$$I = \frac{1}{(x+1)^2} \int \frac{e^x}{(x+1)^2} dx$$

(6) $\int e^x(x+1) dx$

$$u = (x+1)^2$$

$$dv = e^x dx \quad du = 2(x+1)dx$$

$$\int u dv = uv - \int v du$$

$$v = e^x$$

$$I = \int (x+1)^2 e^x dx$$

$$= uv - \int v du$$

$$= (x+1)^2 e^x - \int 2(x+1)e^x dx$$

$$I = (x+1)^2 e^x - 2 \int (x+1)e^x dx$$

To find $\int e^x(x+1) dx$

$$u = x+1$$

$$dv = e^x dx$$

$$du = dx$$

$$\int u dv = uv - \int v du$$

$$v = e^x$$

$$\therefore \int e^x(x+1) dx = uv - \int v du$$

$$= (x+1)e^x - \int e^x dx$$

$$= (x+1)e^x - e^x + c$$

using in (1)

$$I = (x+1)^2 e^x - 2[(x+1)e^x - e^x] + c$$

18.3. Let us sum up

We have studied so far how to evaluate the integral whenever the integral is of the form $\int u dv$

18.4 Check your progress

(7) $\int x^2 e^{x^2} dx$

Hint. $\int \cos^2 x dx = \frac{1 + \cos 2x}{2}$

Take $u=x$; $dv=\sin^2 x dx$

(8) $\int x \cos x^2 dx$

Take $u=x$; $dv=\cos^2 x dx$

(9) $\int x \sin^3 x dx$

Use: $\sin^3 x = 3\sin x - 4\sin^3 x$

$4\sin^3 x = 3\sin x - \sin^3 x$

$\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin^3 x$

(10) $\int x \cos^3 x dx$

use: $\cos^3 x = 4\cos^3 x - 3\cos x$

$4\cos^3 x = \cos^3 x + 3\cos x$

$\therefore \cos^3 x = \frac{1}{4} \cos^3 x + \frac{3}{4} \cos x$

18.5. Lesson end Activities

Integrate the following

(1) $(\log x)^2$ $[x(\log x)^2 - 2\log x + 2]$

(2) $\frac{x \sin^{-1} x}{\sqrt{1-x^2}}$ $[x - \sqrt{1-x^2}] \sin^{-1} x$

(3) $\int_0^1 \frac{x^3 \sin^{-1} x}{\sqrt{1-x^2}} dx$ $[\frac{1}{a}]$

(4) Prove that $\int e^x [f(x) + f'(x)] dx = e^x f(x)$ and hence integrate $(1+x \log x) \frac{e^x}{x}$

(5) $x a^x$ $a^x \left[\frac{x}{\log a} - \frac{1}{(\log a)^2} \right]$

18.6. Points for discussion

(1) Evaluate $\int e^{ax+b} \cos cx dx$ $\frac{e^{ax+b}}{a^2+b^2} [a \cos cx + b \sin cx]$

(2) $\int e^{ax+b} \sin cx dx$ $[\frac{e^{ax+b}}{a^2+b^2} [a \sin cx - b \cos cx]]$

18.7. References

A book of Integral Calculus by T.K. Manickavasagam Pillai

Lesson – 19

REDUCTION FORMULAS

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- 19.0. Aims and Objectives
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19.0 Aims and Objectives

We are going to study about the evaluation of integrals

$\int x^n e^{ax} dx$, $\int x^n \cos ax dx$, $\int x^n \sin ax dx$ using integration by parts.

19.1. Reduction Formula

19.2. Examples

Find a reduction formula for $\int x^n e^{ax} dx$

$$I_n = \int x^n e^{ax} dx$$

Integrate by parts

$$u = x^n \qquad dv = e^{ax} dx$$

$$du = nx^{n-1} dx \qquad v = \frac{e^{ax}}{a}$$

$$v = \frac{e^{ax}}{a}$$

$$I_n = uv - \int v du$$

$$= \frac{x^n e^{ax}}{a} - \int \frac{e^{ax}}{a} \cdot nx^{n-1} dx$$

$$I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$$

The ultimate integral is U_0

$$U_0 = \int e^{ax} dx = \frac{e^{ax}}{a} + c$$

Find a reduction formula for

$$\int x^n \cos ax dx$$

Prove let $I_n = \int x^n \cos ax dx \dots (1)$

In regard by parts

$$u = x^n; dv = \cos ax dx$$

$$dv = \sin ax dx; \quad v = \frac{\sin ax}{a}$$

(1) because

$$\begin{aligned} I_n &= uv - \int v du \\ &= \frac{x^n \sin ax}{a} - \int \frac{\sin ax}{a} n x^{n-1} dx \\ &= \frac{x^n \sin ax}{a} - \frac{n}{a} \int x^{n-1} \sin ax dx \end{aligned}$$

we shall find $\int x^{n-1} \sin ax dx$ again by parts.

$$u = x^{n-1}; dv = \sin ax dx$$

$$dv = \cos ax dx; \quad v = \frac{\cos ax}{a}$$

$$v = \frac{\cos ax}{a}$$

$$\begin{aligned} \int x^{n-1} \sin ax dx &= \frac{\cos ax}{a} x^{n-1} - \int \frac{\cos ax}{a} (n-1)x^{n-2} dx \\ &= \frac{\cos ax}{a} x^{n-1} - \frac{(n-1)}{a} \int x^{n-2} \cos ax dx \\ &= \frac{\cos ax}{a} x^{n-1} - \frac{(n-1)}{a} I_{n-2} \end{aligned}$$

using this in (2)

$$I = \frac{x \sin ax}{a} - \frac{1}{a} \int \frac{\cos ax}{a} dx \quad n-2$$

1) The ultimate integral is I when n is odd.

$$I_1 = \int \cos ax dx$$

Integrate by parts

$$u=x \quad dv = \cos ax dx$$

$$du=dx \quad \int dx = \frac{\sin ax}{a}$$

$$v = \frac{\sin ax}{a}$$

$$I_1 = uv - \int v dx$$

$$= \frac{x \sin ax}{a} - \int \frac{\sin ax}{a} dx$$

$$= \frac{x \sin ax}{a} + \frac{1}{a} \int \cos ax dx$$

$$= \frac{x \sin ax}{a} + \frac{1}{a^2} \left(\frac{\sin ax}{a} \right) + c$$

When n is even, the ultimate integral is I₀

$$I_0 = \int \cos ax dx = \frac{\sin ax}{a} + c$$

(3) Obtain a reduction formula for

$$\int x^n \sin ax dx$$

$$\text{Proof } u_n = \int x^n \sin ax dx$$

Integrate by parts

$$u=x^n ; dv = \sin ax dx$$

$$dv = nx^{n-1} dx$$

$$\int dx = -\frac{\cos ax}{a}$$

$$v = -\frac{\cos ax}{a}$$

$$I_n = uv - \int v dx$$

$$= \frac{-x^n \cos a x}{2} \int \left(\frac{\cos a x}{x} \right)^n x^{n-1} dx$$

$$= \frac{-x^n \cos a x}{a} \int x^{n-1} \cos a x dx \dots (1)$$

To find $\int x^{n-1} \cos ax dx$ by parts

$$u = x^{n-1}; dv = \cos ax dx$$

$$du = (n-1)x^{n-2} dx; \int dv = \int \cos ax dx$$

$$v = \frac{\sin ax}{a}$$

$$\int x^{n-1} \cos ax dx = uv - \int v du$$

$$= \frac{x^{n-1} \sin ax}{a} - \int \frac{\sin ax}{a} (n-1)x^{n-2} dx$$

$$= \frac{x^{n-1} \sin ax}{a} - \frac{(n-1)}{a} \int x^{n-2} \sin ax dx$$

$$= \frac{x^n \cos ax}{a} - \frac{(n-1)}{a} I_{n-2}$$

using in (1)

$$I_n = \frac{-x^n \cos ax}{a} + \frac{\sin ax}{a} \left[\frac{x^n}{n} - \frac{\sin ax}{a} \right] - \frac{(n-1)}{a} I_{n-2}$$

$$= \frac{-x^n \cos ax}{a} + \frac{\sin ax}{a} \left[\frac{x^n}{n} - \frac{\sin ax}{a} \right] - \frac{(n-1)}{a} I_{n-2}$$

i) when n is even, the ultimate integral is I_0

$$\therefore I_0 = \int \cos ax dx = \frac{\sin ax}{a}$$

(ii) when n is odd, the ultimate integral is I_1

$$\therefore I_1 = \int \sin ax dx = -\frac{\cos ax}{a}$$

$$u = x; dv = \sin ax dx$$

$$du = dx; \int dv = \int \sin ax dx$$

$$v = -\frac{\cos ax}{a}$$

$$\begin{aligned}
 I_1 &= \int \frac{-xc + \alpha}{a} \frac{s}{a} dx \\
 &= \frac{-xc + \alpha}{a} \frac{s}{a} \int c \cos x dx \\
 &= \frac{-xc + \alpha}{a} \frac{1}{a} \left(\frac{\sin x}{a} \right) + c \\
 &= \frac{-xc + \alpha}{a} \frac{s}{a^2} \sin x + c
 \end{aligned}$$

(4) Obtain a reduction formula for $\int \sin^n x dx$

$$\begin{aligned}
 \text{Let } I^n &= \int \sin^n x dx \\
 &= \int \sin^{n-1} x \sin x dx
 \end{aligned}$$

Integrate by parts

$$u = \sin^{n-1} x; \quad du = (n-1)\sin^{n-2} x \cos x dx$$

$$dv = \sin x dx; \quad v = -\cos x$$

$$v = -\cos x$$

$$\text{Let } u_n = \int \sin^n x dx$$

$$= u \cdot v - \int v \cdot du$$

$$= \sin^{n-1} x (-\cos x) - \int (-\cos x) (n-1)\sin^{n-2} x dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos x dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx$$

$$u_n = -\sin^{n-1} x \cos x + (n-1) u_{n-2}$$

$$u_n + (n-1) u_{n-2} = -\sin^{n-1} x \cos x + (n-1) u_{n-2}$$

$$u_n - (n-1) u_{n-2} = -\sin^{n-1} x \cos x$$

$$u_n = -\sin^{n-1} x \cos x + (n-1) u_{n-2}$$

$$u_n = \frac{1}{n} \sin^{n-1} x \cos x + \frac{n-1}{n} u_{n-2} \dots (1)$$

i) When n is even, the ultimate integral is U_0

$$U_0 = \int dx = x + c$$

ii) When n is odd, the ultimate integral is U_1

$$U_1 = \int_0^{\pi/2} \sin x \, dx$$

5) Find a reduction formula for $\int_0^{\pi/2} \sin^n x \, dx$

Proof :

$$U_n = \int_0^{\pi/2} \sin^n x \, dx$$

$$U_n = \frac{n-1}{n} U_{n-2} \quad \therefore \text{the first term vanishes at both the limits}$$

$$= \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \dots U_1 \text{ if } n \text{ is odd}$$

$$U_1 = \int_0^{\pi/2} \sin x \, dx$$

$$\therefore U_n = \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \dots \text{ if } n \text{ is odd}$$

If n is even

$$U_n = \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \dots U_0$$

$$\therefore U_0 = \int_0^{\pi/2} \sin^0 x \, dx = \int_0^{\pi/2} 1 \, dx = \frac{\pi}{2}$$

$$\therefore \text{If } n \text{ is even, } U_n = \frac{n-1}{n} \frac{n-3}{n-2} \frac{n-5}{n-4} \dots \frac{\pi}{2}$$

(6) Find a reduction formula for $\int_0^{\pi/2} \cos^n x \, dx$

Proof : Let $U_n = \int_0^{\pi/2} \cos^n x \, dx$

$$= \int_0^{\pi/2} \cos^{n-1} x \cos x \, dx$$

Integrate by parts

$$u = \cos^{n-1} x; \, dv = \cos x \, dx$$

$$du = -(n-1)\cos^{n-2} x \sin x \, dx; \, \int dv = \int_0^{\pi/2} \cos x \, dx$$

$$v = \sin x$$

$$U_n = U_v - \int v \, du$$

$$= \sin x \cos^{n-1} x - \int_0^{\pi/2} (1-x^2) \cos x \, dx$$

$$\begin{aligned}
 &= \int_0^{\pi/2} c \cos^n x \, dx \\
 &= \int_0^{\pi/2} c \cos^{n-1} x \cos x \, dx \\
 &= \int_0^{\pi/2} c \cos^{n-1} x \sin x \, dx \\
 &\therefore U_n = \int_0^{\pi/2} c \cos^{n-1} x \sin x \, dx \\
 &= \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} - \frac{c}{n} \int_0^{\pi/2} \cos^{n-2} x \sin^2 x \, dx \\
 &= \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} - \frac{c}{n} \int_0^{\pi/2} \cos^{n-2} x (1 - \cos^2 x) \, dx \\
 &= \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} - \frac{c}{n} \int_0^{\pi/2} \cos^{n-2} x \, dx + \frac{c}{n} \int_0^{\pi/2} \cos^n x \, dx \\
 &= \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} - \frac{c}{n} U_{n-2} + \frac{c}{n} U_n \\
 &\therefore U_n = \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} - \frac{c}{n} U_{n-2} + \frac{c}{n} U_n \\
 &\therefore U_n = \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} + \frac{c}{n} U_n \\
 &\therefore U_n = \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} + \frac{c}{n} U_n \\
 &\therefore U_n = \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} + \frac{c}{n} U_n
 \end{aligned}$$

If n is odd, the ultimate integral is U_1

$$U_1 = \int_0^{\pi/2} c \cos x \, dx = c \sin x \Big|_0^{\pi/2} = c$$

If n is even, the ultimate integral is U_0

$$U_0 = \int_0^{\pi/2} c \, dx = cx \Big|_0^{\pi/2} = \frac{c\pi}{2}$$

(7) Find a reduction formula for $\int_0^{\pi/2} c \cos^n x \, dx$

Proof : (Hint : Write upto (1) from Book work (6))

$$\begin{aligned}
 U_n &= \int_0^{\pi/2} c \cos^n x \, dx \\
 &= \int_0^{\pi/2} c \cos^{n-1} x \cos x \, dx \\
 &= \int_0^{\pi/2} c \cos^{n-1} x \sin x \, dx \\
 &= \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} - \frac{c}{n} \int_0^{\pi/2} \cos^{n-2} x \sin^2 x \, dx \\
 &= \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} - \frac{c}{n} \int_0^{\pi/2} \cos^{n-2} x (1 - \cos^2 x) \, dx \\
 &= \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} - \frac{c}{n} \int_0^{\pi/2} \cos^{n-2} x \, dx + \frac{c}{n} \int_0^{\pi/2} \cos^n x \, dx \\
 &= \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} - \frac{c}{n} U_{n-2} + \frac{c}{n} U_n \\
 &\therefore U_n = \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} - \frac{c}{n} U_{n-2} + \frac{c}{n} U_n \\
 &\therefore U_n = \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} + \frac{c}{n} U_n \\
 &\therefore U_n = \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} + \frac{c}{n} U_n
 \end{aligned}$$

\therefore when n is odd

$$U_n = \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} + \frac{c}{n} U_n$$

When n is even, the ultimate integral is U_0

$$\begin{aligned}
 U_0 &= \int_0^{\pi/2} c \, dx = \frac{c\pi}{2} \\
 \therefore U_n &= \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} + \frac{c}{n} U_n \\
 &= \frac{c}{n} \cos^{n-1} x \sin x \Big|_0^{\pi/2} + \frac{c}{n} U_n
 \end{aligned}$$

Examples for Book works 5 and 7

$$(1) \int_0^{\pi/2} \sin^8 x \, dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1\pi}{2}$$

$$(2) \int_0^{\pi/2} \sin^5 x \, dx = \frac{4}{5} \cdot \frac{2}{3} \cdot 1 \cdot \frac{8}{1}$$

$$(3) \int_0^{\pi/2} \sin^8 x \, dx = \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1\pi}{2}$$

$$(4) \int_0^{\pi/2} \sin^5 x \, dx = \frac{4}{5} \cdot \frac{2}{3} \cdot 1 \cdot \frac{8}{1}$$

Book work

(8) Find a reduction formula for

$$\int \sin^m x \cos^n x \, dx$$

$$\begin{aligned} \text{Proof } U_{m,n} &= \int \sin^m x \cos^n x \, dx \\ &= \int \sin^m x \cos^{n-1} x \cos x \, dx \end{aligned}$$

Integrate by parts.

$$u = \cos^{n-1} x$$

$$du = -(n-1)\cos^{n-2} x \sin x \, dx$$

$$\int dx \Rightarrow \int \sin^m x \cos^n x \, dx$$

Put $\sin x = t$

$$\cos x \, dx = dt$$

$$\therefore v = \int u \, dv = \frac{t^{m+1}}{m+1} = \frac{\sin^{m+1} x}{m+1}$$

$$U_{m,n} = uv - \int v \, du$$

$$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} - \int \frac{\sin^{m+1} x}{m+1} \cdot (-n) \cos^{n-2} x \, dx$$

$$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{n}{m+1} \int \sin^{m+1} x \cos^{n-2} x \, dx$$

$$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{n}{m+1} \int \cos^n x \sin^2 x \, dx$$

$$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} + \frac{n}{m+1} \int \cos^n x \sin^2 x \, dx$$

$$= c \int_0^{\pi/2} x^{m+1} \sin^{n-1} x \cos^{m+1} x \, dx$$

$$U_{m,n} = c \int_0^{\pi/2} x^{m+1} \sin^{n-1} x \cos^{m+1} x \, dx$$

$$U_{m,n} = c x^{n-1} \int_0^{\pi/2} \frac{\sin^{m+1} x}{m+1} \frac{\cos^{n-1} x}{n-1} \, dx$$

$$U_{m,n} + \frac{n-1}{m+1} U_{m,n-1} = c \int_0^{\pi/2} \frac{\sin^{m+1} x}{m+1} \cos^{n-1} x \, dx$$

$$\frac{n-1}{m+1} U_{m,n} = c \int_0^{\pi/2} \frac{\sin^{m+1} x}{m+1} \cos^{n-1} x \, dx$$

$$U_{m,n} = \frac{1}{m+1} c \int_0^{\pi/2} \sin^{m+1} x \cos^{n-1} x \, dx$$

If n is odd, the ultimate integral is $U_{m,1}$

If n is even, the ultimate integral is $U_{m,0}$

Book work 9 obtain a reduction formula for

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$\text{Proof } U_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

From Book work No:8, write upto the result (1)

$$U_{m,n} = \frac{1}{m+1} c \int_0^{\pi/2} \sin^{m+1} x \cos^{n-1} x \, dx$$

$$= \frac{n-1}{m+n} U_{m,n-2}; \text{ the first term vanishes at both the limits}$$

similarly

$$U_{m,n} = \frac{n-3}{m+n-2} U_{m,n-4}$$

$$U_{m,n} = \frac{n-5}{m+n-4} U_{m,n-6}$$

$$U_{m,n} = \frac{n-7}{m+n-6} U_{m,n-8}$$

$$U_{m,n} = \frac{n-1}{m+n} \frac{n-3}{m+n-2} \frac{n-5}{m+n-4} \dots \text{ when}$$

n is odd $U_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$

put $\sin x = t$ when $x=0, t=0$

$\cos x dx = dt$ $x = \frac{\pi}{2} t=1$

$$= \int_0^1 t^m dt = \left[\frac{t^{m+1}}{m+1} \right]_0^1 = \frac{1}{m+1}$$

\therefore when n is odd,

$$U_{m,n} = \frac{n-1}{m+n} \times \frac{n-3}{m+n-2} \times \frac{n-5}{m+n-4} \times \dots \times \frac{1}{m+1}$$

when n is even, the ultimate integral is $U_{m,0}$

$$U_{m,0} = \int_0^{\pi/2} \sin^m x dx$$

$$= \frac{m-1}{m} \times \frac{m-3}{m-2} \times \frac{m-5}{m-4} \times \dots \text{ is } m \text{ is odd}$$

$$= \frac{m-1}{m} \times \frac{m-3}{m-2} \times \frac{m-5}{m-4} \times \dots \times \frac{\pi}{2} \text{ is } m \text{ is even}$$

Book work No :10

Find a reduction formula for $\int t^n dx$

$$U_n = \int t^n dx$$

$$= \int t^{n-2} \cdot t^2 dx$$

$$= \int t^{n-2} (x - \frac{2}{3} x^3 + \frac{1}{5} x^5) dx$$

$$= \int t^{n-2} x dx - \frac{2}{3} \int t^{n-2} x^3 dx + \frac{1}{5} \int t^{n-2} x^5 dx$$

Put $\tan x = t$

$\sec^2 x dx = dt$ in first term

$$U_n = \int t^{n-2} dt = U_{n-2}$$

$$= \frac{t^{n-1}}{n-1} = U_{n-2}$$

If n is odd, the ultimate integral is U_1

$$U_1 = \int t^0 dx = x + c$$

If n is even, the ultimate integral is U_0

$$U_0 = \int t^0 dx = x + c$$

Book work No :11

Find a reduction formula for $\int \cos^n x dx$

$$\begin{aligned}
 U_n &= \int \cos^n x dx \\
 &= \int \cos^{n-2} x \cos^2 x dx \\
 &= \int \cos^{n-2} x (-\sin x \cos x)' dx \\
 U_n &= \int \cos^{n-2} x dx - \int \cos^{n-2} x \sin^2 x dx
 \end{aligned}$$

Put $\cos x = t$ in first integral

$$-\cos^2 x dx = dt$$

$$\cos^2 x dx = -dt$$

$$\begin{aligned}
 U_n &= \int t^{n-2} dt - U_{n-2} \\
 &= \frac{t^{n-1}}{n-1} - U_{n-2} \\
 &= \frac{1}{n-1} \cos^{n-1} x - U_{n-2}
 \end{aligned}$$

If n is odd, the ultimate integral is U_1

$$\begin{aligned}
 U_1 &= \int \cos x dx \\
 &= \log \sin x + c
 \end{aligned}$$

If n is even, the ultimate integral is U_0

$$U_0 = \int 1 dx = x + c$$

Book work No :12

find a reduction formula for $\int \sec^n x dx$

$$\begin{aligned}
 U_n &= \int \sec^n x dx \\
 &= \int \sec^{n-2} x \sec^2 x dx
 \end{aligned}$$

Integral by parts

$$\begin{aligned}
 U &= \sec^{n-2} x \\
 dx &= (n-2) \sec^{n-3} x \sec x \tan x dx \\
 &= (n-2) \sec^{n-2} x \sec x \tan x dx \\
 dv &= \sec^2 x dx
 \end{aligned}$$

$$\int d \Rightarrow \int s x e^2 d$$

$$v = \tan x$$

$$\therefore U_n = u_v \int d$$

$$= s e^{n-2} x - a n \int x^{n-2} (2) x s e x c t d a n x$$

$$= s e^{n-2} x t a n - (n-2) \int x e^2 x (a d n) x$$

$$= s e^{n-2} x t - a x n^2 (n-2) \int x^2 e x (-s e c) x$$

$$= s e^{n-2} x t x a - n^2 (2x) \int s e^{n-2} x s x e d x$$

$$= s e^{n-2} x t x a - n^2 (2x) \int s e^{n-2} x s x e d x$$

$$= s e^{n-2} x t x a - n^2 (2x) \int s e^{n-2} x s x e d x$$

$$U_n + (n-2) v s_n e^{n-2} t x a - n^2 (2x) \int s e^{n-2} x s x e d x$$

$$(n-1) \int s e^{n-2} t x a - n^2 (2x) \int s e^{n-2} x s x e d x$$

$$U_n = \frac{1}{n-1} s e^{n-2} t x a - \frac{2}{n-1} n \int s e^{n-2} x s x e d x$$

When n is odd, the ultimate integral is U₁

$$U_1 = \int s e^{n-2} t x a - n^2 (2x) \int s e^{n-2} x s x e d x$$

when n is even, the ultimate integral is U₀

$$U_0 = \int s e^{n-2} t x a - n^2 (2x) \int s e^{n-2} x s x e d x$$

Book work 13

Obtain a reduction formula for $\int c o s e^x x d$

proof $U_n = \int c o s x^n d$

$$= \int c o s^{n-2} x e^2 c s x d$$

Integral by parts

$$u = c o s^{n-2} x$$

$$du = -(n-2) c o s e c^{n-3} x c o s e c x c o t x d x$$

$$= -(n-2) c o s e c^{n-2} x c o t x d x$$

$$dv = \operatorname{cosec}^2 x dx$$

$$\int d \neq \int c o s^2 d$$

$$v = -\cot x$$

$$\therefore u_n \neq v - \int d$$

$$= -\cot x \operatorname{cosec} x - 2x - \int (x-2) \operatorname{cosec}^2 x dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (x-2) \int \operatorname{cosec}^2 x dx$$

$$= -\cot x \operatorname{cosec} x - 2x - (x-2) \left[\int \operatorname{cosec}^2 x dx \right]$$

$$= -\cot x \operatorname{cosec}^{x-2} x - (x-2)(u_x - u_{x-2})$$

$$= -\cot x \operatorname{cosec}^{x-2} x - (x-2)u_x + (x-2)u_{x-2}$$

$$\therefore u_n = (n-2)u_{n-2} - \cot x \operatorname{cosec}^{n-2} x - (n-2)u_n$$

$$(x-1)u_x = -\cot x \operatorname{cosec}^{x-2} x + (x-2)u_{x-2}$$

$$u_x = -\frac{1}{x-1} \cot x \operatorname{cosec}^{x-2} x + \frac{x-2}{x-1} u_{x-2}$$

If x is even, the ultimate integral is u_0

$$\therefore u_0 = \int dx = x + c$$

If x is odd, the ultimate integral is u_1

$$u_1 = \int \operatorname{cosec} x dx$$

$$= -\log(\operatorname{cosec} x + \cot x) + c$$

19.3 Let us sum up

Using integration by parts we have studied evaluation of the integrals $\int_0^{\pi/2} \sin^n x \, dx$ and $\int_0^{\pi/2} \cos^n x \, dx$.

19.4 Check your Progress

(1) Evaluate $\int_0^{\pi/2} \sin^6 x \, dx$

(2) $\int_0^{\pi/2} \cos^7 x \, dx$

(3) $\int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$

(4) $\int_0^{\pi/2} \sin^3 x \cos^3 x \, dx$

19.5 Lesson end activities

1. If $u_x = \int \frac{dx}{(x^2+1)^x}$, show that

$$2x \cdot u_x + 1 = \frac{x}{(x^2+1)^x} + (2x-1)u_x$$

2. If $u_x = \int_0^a (a^2 - x^2)^x \, dx$ prove that $u_x = \frac{2x-2}{2x+1} u_{x-1} - 1$

Hence obtain a reduction formula for $\int_0^1 (1-x^2)^m \, dx$ where m is a positive integer

3. If $u_x = \int_{-1}^1 (1-x^2)^x \, dx$ prove that

$$u_x = \frac{2x}{2x+1} u_{x-1}$$

19.6 Points for discussion

4. If $u_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$ show that

$$u_{m,n} = \frac{\cos^m x \sin^{n+1} x}{m+n} + \frac{m}{m+n} u_{m-1,n}$$

5. If $u_{m,x} = \int_0^{\pi/2} \cos^m x \sin^x \, dx$ prove that

$$u_{m,x} = \frac{m}{m+x} u_{m-1,x}$$

Hence deduce $u_{nx} = \frac{\pi}{2^{n+1}}$

19.7. References

A book of Integral Calculus by T.K. Manickavasagam Pillai

Lesson – 20

Double and Triple Integrals

Contents

- 20.0. Aims and Objectives
- 20.1. Double Integrals
- 20.2. Examples
- 20.3. Triple Integrals
- 20.4. Examples
- 20.5. Let us sum up
- 20.6. Check your progress
- 20.7. Lesson end activities
- 20.8. Points for Discussion
- 20.9. References

20.0 Aims and Objectives

We shall study in this lesson how to evaluate double and triple integrals.

$\int_a^b \int_c^d f(x,y) dx dy$ using integration by parts.

20.1 Double Integrals:

When a function $f(x)$ is integrated with respect to x between the limits a and b , we have the definite integral $\int_a^b f(x) dx$. If the integral is a function x and y i.e., $f(x,y)$ and if it is integrated between the limits a and b for x and c and y

for y , we get a double integral and is denoted by the symbol

$$\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y) dx dy$$

$y=c \quad x=a$

Evaluation of Double integrals

To evaluate $\int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x,y) dx dy$, we first integrate $f(x,y)$ with respect

to x partially (treating y , a constant), between a and b . After this inner integration and substituting the limits for x , we get the integrand a function of y . If we integrate this integrand, w.r.t. y between the limits $y=c$ and $y=d$,

we get the double integral $\int_c^d \int_a^b f(x,y) dx dy$ evaluated. The following is the

illustration of a double integral to be evaluated.

$$\int_c^d \int_a^b f(x,y) dx dy$$

20.2 Examples :

$$1. \int_0^2 \int_0^1 4xy dx dy$$

$$I = \int_0^2 \int_0^1 4xy dx dy$$

$$= 4 \int_0^2 \left[\frac{x^2}{2} dy \right]_0^1$$

$$= 4 \int_0^2 \left[\frac{y^2}{2} \right]_0^1 dy$$

$$= 2 \int_0^2 dy$$

$$= 2(y)_0^2 = 2(2-0) = 4$$

$$2. \int_0^{\pi} \int_0^{\sin \theta} r dr d\theta$$

$$\begin{aligned}
 &= \int_0^{\pi} \left[\int_0^{2 \cos \theta} f(r) r dr \right] d\theta \\
 &= \int_0^{\pi} \left[\frac{r^2}{2} \right]_0^{2 \cos \theta} d\theta \\
 &= \frac{1}{2} \int_0^{\pi} [4 \cos^2 \theta] d\theta \\
 &= \frac{1}{4} \left[\int_0^{\pi} d\theta - \int_0^{\pi} \cos 2\theta d\theta \right] \\
 &= \frac{1}{4} \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi} \\
 &= \frac{1}{4} \left[\pi - \frac{1}{2} (\sin 2\pi - \sin 0) \right] \\
 &= \frac{\pi}{4}
 \end{aligned}$$

20.3 Triple Integrals

Extending the concept of the double integral to one more variable

z, we get the triple integral $\int_{z=e}^{z=f} \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y, z) dx dy dz$

The following is the evaluation chart for a triple integral.

$$\int_{z=e}^{z=f} \int_{y=c}^{y=d} \int_{x=a}^{x=b} f(x, y, z) dx dy dz$$

20.4 Examples

- $\int_0^1 \int_0^2 \int_0^3 xyz dx dy dz$

$$\begin{aligned}
 I &= \int_0^1 \int_0^2 \int_0^3 xyz \, dx \, dy \, dz \\
 &= \int_0^1 \left(\int_0^2 \left(\int_0^3 x \, dx \right) y \, dz \right) dy \\
 &= \int_0^1 \left(\frac{x^2}{2} \Big|_0^3 \right) y \, dz \, dy \\
 &= \frac{9}{2} \int_0^1 \left(\int_0^2 y \, dz \right) dy \\
 &= \frac{9}{2} \int_0^1 \left(\frac{y^2}{2} \Big|_0^2 \right) dy \\
 &= 9 \int_0^1 z \, dz = 9 \left(\frac{z^2}{2} \Big|_0^1 \right) = \frac{9}{2} (1 - 0) = \frac{9}{2}
 \end{aligned}$$

2. $\int_0^a \int_0^b \int_0^c (x + y + z) \, dx \, dy \, dz$

$$\begin{aligned}
 &= \int_0^a \left(\int_0^b \left(\int_0^c (x + y + z) \, dz \right) dy \right) dx \\
 &= \int_0^a \left[\int_0^b \left[x + y + \frac{z^2}{2} \Big|_0^c \right] dy \right] dx \\
 &= \int_0^a \left[\int_0^b \left(x + y + \frac{c^2}{2} \right) dy \right] dx \\
 &= \int_0^a \left[x + \frac{y^2}{2} + \frac{c^2}{2} \right] \Big|_0^b dx \\
 &= \int_0^a \left[x + \frac{b^2}{2} + \frac{c^2}{2} \right] dx
 \end{aligned}$$

$$\begin{aligned}
 &= \left[\frac{x^2}{2} b + \frac{c}{2} \right]_0^a \frac{b}{x} \\
 &= \frac{a^2 b}{2} + \frac{c}{2} b \\
 &= \frac{a b}{2} (a + b + c)
 \end{aligned}$$

20.5 Let us sum up

So far we have learnt an evaluation of double and triple Integrals.

20.6 Check your progress

1. Evaluate $\int_0^1 \int_0^1 x y \, dy \, dx$
2. Evaluate $\int_0^1 \int_0^1 \int_0^1 x y z \, dz \, dy \, dx$

20.7 Lesson end activities

Ans:

1. Evaluate $\int_1^b \frac{d}{x} \frac{dx}{y}$ $\log_a \log_b$
2. Evaluate $\int_0^{\pi/2} \int_0^{\pi/2} \sin(\theta + \phi) \, d\theta \, d\phi$ 2
3. Evaluate $\int_0^1 \int_0^{y+z} \int_0^z d \, dx \, dy \, dz$ $\frac{1}{2}$
4. Prove that $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} d \, dz \, dy \, dx = \frac{\pi a^3}{6}$
5. Prove that $\int_0^1 \int_0^{1-n} \int_0^{1-n-y} x y z \, dx \, dy \, dz = \frac{1}{72}$

6. Show that $\int_0^1 \int_0^{2-x} \int_0^{2-x-y} (x+y+z) dz dy dx = \frac{8}{3}$

20.8 Points for Discussion

7. Evaluate $\int xy dx$ over the region in the positive Quadrant by the line

$2x+3y=6$ (Ans. 3/2)

8. Evaluate $\int \int (x+y+z) dx dy dz$ where v is the region bonded by $x=0,$

$x=1, y=0, y=2, z=0, z=3$ (Ans. 33/2)

20.9 References

A book of Integral Calculus by T.K. Manickavasagam Pillai

Lesson – 21

AREAS AND VOLUMES

Contents

- 21.0. Aims and Objectives
- 21.1. Applications to Calculations of areas and volumes
- 21.2. Examples
- 21.3. Areas in Polar Coordinates
- 21.4. Examples
- 21.5. Let us sum up
- 21.6. Check your Progress
- 21.7. Lesson end activities
- 21.8. Points for discussion
- 21.9. References

21.0 Aims and Objectives

We shall study on the applications of to calculations of areas and volumes.

21.1 Applications to Calculations of areas and volumes

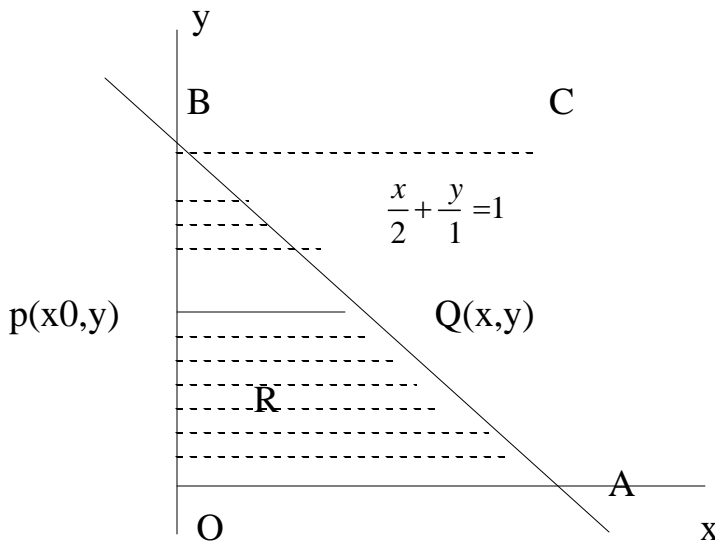
21.2 Examples

1. Evaluate $\int x y \, dx$, where R is the region bounded by the R line $x+2y=2$

lying in the first quadrant

Proof : The equation of the line is $x+2y=2$

$$\therefore \frac{x}{2} + \frac{y}{1} = 1$$



R is the region as show here.

The limits of the variables of integration are not given. \therefore we can choose the order of integration arbitrarily.

We shall integrate first w.r.t. to x and then w.r.t y.

Treating $y=a$ constant means that it is equivalent to drawing a line \parallel to x axis arbitrarily lying in the region R of integration. Let P be (x_0,y) ; Q be (x_1,y) , since PQ is \parallel to x axis ($y=a$ constant)

x varies from x_0 to x_1

$$\therefore x_0=0 \text{ and } x_1+2y=2$$

$$\therefore (x_1, y) \text{ lies on } x+2y=2$$

$$\therefore x_1+2y=2$$

$$x_1=2-2y$$

$$\therefore \text{The limits for x are } x=0 \text{ to } x=2-2y$$

To find the limits for y :

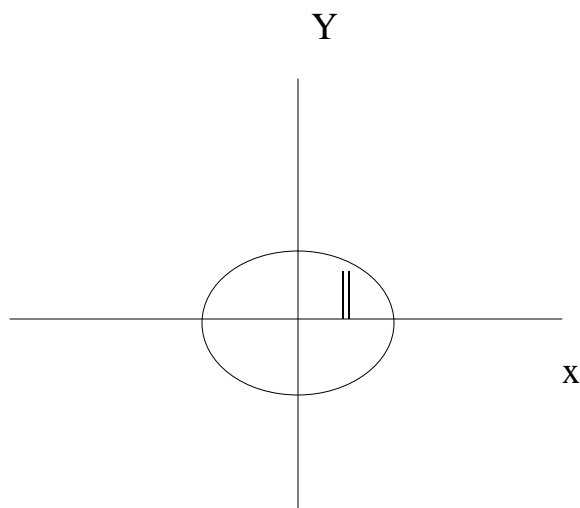
We have to find the variation of the line PQ. So that the region R is fully covered. To sweep the entire area of R, PQ has to start from the position where $y=0$, more parallel to itself and should go up to the position BC where $y=1$.

$$\therefore y=0 \text{ to } y=1$$

$$\begin{aligned} I &= \int_0^1 \int_0^{1-2y} y \, dx \, dy \\ &= \int_0^1 y \left(\frac{x^2}{2} \right)_0^{1-2y} dy \\ &= \frac{1}{2} \int_0^1 y(1-2y)^2 dy \\ &= 2 \int_0^1 y(1-2y) dy \\ &= 2 \int_0^1 y(1-2y) dy \\ &= 2 \int_0^1 (y^3 + y^2 - 2y) dy \\ &= 2 \left[\int_0^1 y^3 dy + \int_0^1 y^2 dy - 2 \int_0^1 y dy \right] \\ &= 2 \left[\frac{y^4}{4} + \frac{y^3}{3} - 2 \frac{y^2}{2} \right]_0^1 \\ &= \frac{1}{6} \end{aligned}$$

2. Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by double integration.

Solution :



Area of the Ellipse = 4 x area of the quadrant of the ellipse

$$= 4 \times \int dy \, dx$$

$$= 4 \int_{x=0}^a \int_{y=0}^{b\sqrt{1-\frac{x^2}{a^2}}} dy \, dx$$

$$= 4 \int_{x=0}^a [y]_0^{b\sqrt{1-\frac{x^2}{a^2}}} dx$$

$$= 4b \int_0^a \sqrt{\frac{a^2 - x^2}{a^2}} dx$$

$$= \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} dx$$

$$= \frac{4b}{a} \left\{ \frac{2a}{2} \sin^{-1} \left(\frac{x}{a} \right) + \frac{\sqrt{a^2 - x^2}}{2} \right\}_0^a$$

$$= \frac{4b \pi a}{4} = \pi a b$$

3. Find the volume of the tetrahedron bounded by the coordinate planes

and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Solution :

$$\text{Volume} = \int \int dx dy dz$$

Evaluation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

i.e., find the limits for x :

Put $y=0, z=0$

$\therefore x=0$ to $x=a$

To find the limits for y

Put $z=0$ $\frac{x}{a} + \frac{y}{b} = 1$

$y = b(1 - x/a)$

To find the limits for z

$$\frac{z}{c} = 1 - \frac{x}{a} - \frac{y}{b}$$

$$z = c(1 - \frac{x}{a} - \frac{y}{b})$$

$$z=0 \text{ to } z=c(1 - \frac{x}{a} - \frac{y}{b})$$

$$V = \int_{x=0}^a \int_{y=0}^{b(1-x/a)} \int_{z=0}^{c(1-x/a-y/b)} dz dy dx$$

$$= \int_{x=0}^a \int_{y=0}^{b(1-x/a)} (z) c(1-x/a-y/b) dy dx$$

$$\begin{aligned}
 &= c \int_{x=0}^a \int_0^{b(1-x/a)} \left[1 - \frac{x}{a} - \frac{y}{b}\right] dy dx \\
 &= c \int_0^a \left[b \left(1 - \frac{x}{a}\right) \right] \frac{y^2}{2} \Big|_0^{b(1-x/a)} dx \\
 &= c \int_0^a b \left(1 - \frac{x}{a}\right) \left(1 - \frac{dx}{a}\right) \frac{1}{2} \frac{b^2}{b} \left(\frac{x}{a}\right)^2 \\
 &= c \int_0^a \left[b \left(1 - \frac{x}{a}\right)^2 - \frac{1}{2} \frac{b^2}{a} \left(\frac{x}{a}\right)^2 \right] dx \\
 &= \frac{b}{2} \int_0^a \left(1 - \frac{x}{a}\right)^2 dx \\
 &= \frac{b}{2} \left[\frac{\left(1 - \frac{x}{a}\right)^3}{-3/a} \right]_0^a \\
 &= \frac{a b}{6} \left[\left(1 - \frac{x}{a}\right)^3 \right]_0^a \\
 &= -\frac{a b}{6} \left\{ \frac{1}{3} \right\} \\
 &= \frac{a b}{6}
 \end{aligned}$$

21.3 Areas in Polar Coordinates

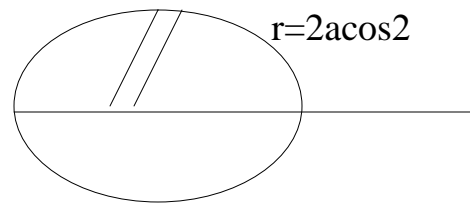
Consider the equation of the curve $r=f(\theta)$

Then area = $\int r^2 d\theta$ with suitable limits for r and θ

21.4 Examples

1. Find the area of the curve $r=a(1+\cos\theta)$ by double integration

Proof: Area = $\int r^2 d\theta$



= 2 x upper area of the diagram shown

$$= 2 \int_0^{\pi} \int_0^{a(1+\cos\theta)} r \, dr \, d\theta$$

$$\left\{ \begin{array}{l} \theta = 0 \text{ to } \pi \\ r = 0 \text{ to } a(1+\cos\theta) \end{array} \right.$$

$$= 2 \int_0^{\pi} \left[\frac{r^2}{2} \right]_0^{a(1+\cos\theta)} d\theta$$

$$= 2 \int_0^{\pi} \frac{a^2}{2} (1+\cos\theta)^2 d\theta$$

$$= a^2 \int_0^{\pi} (2 + 2\cos\theta) d\theta$$

$$= 4a^2 \int_0^{\pi} \cos^2\left(\frac{\theta}{2}\right) d\theta$$

Put $\frac{\theta}{2} = Q \therefore \theta = 2Q$
 $d\theta = 2 \, dQ$

$\theta = 0 \Rightarrow Q = 0$

$\theta = \pi \Rightarrow Q = \frac{\pi}{2}$

$\therefore \text{Area} = 4a^2 \int_0^{\pi/2} \cos^2 Q \cdot 2 \, dQ$

$$= 8a^2 \int_0^{\pi/2} \cos^2 Q \, dQ$$

$$= 8a^2 \times \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} d^3$$

$$\text{Area} = \frac{3\pi}{2} a^2$$

2. Find the area of the circle by double integration, using polar coordinates.

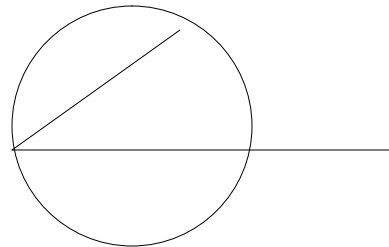
Solution :

We shall take the equation of the circle of radius 'a' as $r = 2a \cos \theta$

Area = 2 x upper area

$$\theta = 0 \text{ to } \frac{\pi}{2}$$

$$r = 0 \text{ to } 2a \cos \theta$$



$$= 2 \int_0^{\pi/2} \int_0^{2a \cos \theta} r \, dr \, d\theta$$

$$= 2 \int_0^{\pi/2} \left[\frac{r^2}{2} \right]_0^{2a \cos \theta} d\theta$$

$$= 4a^2 \int_0^{\pi/2} \cos^2 \theta \, d\theta$$

$$= 4a^2 \times \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \pi a^2$$

21.5 Let us sum up

We have learnt the evaluation of integrals for Areas and volumes.

21.6 Check your Progress

(1) Find the area of the cardioid $r = a(1 + \cos 2\theta)$

21.7 Lesson end activities

1. Prove that the area of the curve $r^2 = a^2 \cos 2\theta$ by double integration is a^2

21.8 Points for discussion

1. Find the volume of the sphere of radius a by triple integral (Ans. $\frac{4\pi a^3}{3}$)

21.9. References

A book of Integral Calculus by T.K. Manickavasagam Pillai

UNIT 5

Lesson – 22

IMPROPER INTEGRALS

Contents

- 22.0 Aims and Objectives
- 22.1 Notion of Improper integrals
- 22.2 Beta Gamma Integrals
- 22.3 Properties of Beta function
- 22.4 Let us sum up
- 22.5 Check your Progress
- 22.6 Lesson end Activities
- 22.7 Points for discussion
- 22.8 References

22.0 Aims and Objectives

22.1 Notion of Improper integrals

Consider the integral $\int_a^b f(x) dx$ $x \leq a$ x

There are three types of integrals

(i). first kind (ii). Second kind (iii). Third kind

(i) first kind of either $a \rightarrow +\infty$ or $b \rightarrow -\infty$ or both, then the case is called an improper integral of first kind.

(ii) second kind : an improper integral is called a second kind if $f(x)$ is unbounded at one or more points of the integral of integration.

(iii) third kind : An improper integral is of third kind if it is of first kind and second kind.

22.2. Beta Gamma Integrals

Definition 1 : the Gamma function

The integral $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$ for $n > 0$ is called a Gamma function.

Definition 2 : The Beta function

The Beta function with parameters m, n is defined by

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad \text{for } m, n > 0$$

Note : 1. The Gamma function is a function of n and is convergent for $n > 0$

2. the Beta function is convergent if $m, n > 0$

22.1.2 Recurrence formula for $\Gamma(n)$

Prove that $\Gamma(n+1) = n\Gamma(n)$ if $n > 0$

Proof : $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1} dx$ (by definition)

$$\therefore \Gamma(n+1) = \int_0^{\infty} e^{-x} x^n dx$$

Integrate by parts

$$u = x^n; \quad dv = e^{-x} dx$$

$$du = nx^{n-1} dx \quad \int dv = \int e^{-x} dx$$

$$v = e^{-x}$$

$$\therefore (x+1)^n = \int_0^\infty e^{-x} x^n dx$$

$$= -n e^{-x} x^{n-1} \Big|_0^\infty + \int_0^\infty e^{-x} x^{n-1} dx$$

$$= 0 + \int_0^\infty e^{-x} x^{n-1} dx \quad \text{the first term vanishes at both the limits}$$

$$= x \cdot (x)$$

$$\therefore (x+1)^n = x(x)$$

Note : 1. $(x+1)^n = x(x)$ if x is a positive integer.

Proof : We have $(x+0) = x(x)$

$$= x(x+1) - (x-1)$$

$$= x(x-1)(x-2) - (x-2)$$

$$= x(x-1)(x-2) \dots - 3.2.1 \quad (1)$$

But $(1) = \int_0^\infty e^{-x} dx$

$$= -e^{-x} \Big|_0^\infty = (0 - 1) = -1$$

$$\therefore (x+1)^n = x(x-1)(x-2) \dots - 3.2.1$$

$$(x+1)^n = x(x)$$

22.3 Properties of Beta function

1. Prove that $B(m, n) = B(n, m)$

Proof : $B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

By using : $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

$$\begin{aligned} \therefore B(m, n) &= \int_0^1 (x^{m-1} - x^{m-1}d)^{n-1} x \, dx \\ &= \int_0^1 (x^{m-1})^n (1-d)^{n-1} x \, dx \\ &= \int_0^1 x^{m-1} (x^{m-1}d)^{n-1} x \, dx \\ &= B(n, m) \end{aligned}$$

2. Prove that $B(m, n) = \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy$

Proof : $B(m, n) = \int_0^1 n^{m-1} (n-x)^{n-1} dx \dots \dots \dots (1)$ for $m, n > 0$

Put $n = \frac{y}{1+y}$ in ... (1)

$$\therefore dx = \frac{(1+y) - y}{(1+y)^2} dy$$

$$\begin{aligned} \therefore 1-n &= 1 - \frac{y}{1+y} \\ &= \frac{1}{1+y} \\ &= \frac{1+y-y}{1+y} \\ &= \frac{1}{1+y} \text{ when } n=0, y=0, n=1, y=\infty \end{aligned}$$

Using in (1)

$$B(m, n) = \int_0^{\infty} \left(\frac{y}{1+y} \right)^{m-1} \left(\frac{1}{1+y} \right)^{n-1} \frac{1}{(1+y)^2} dy$$

$$= \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n+2}} dy$$

$$= \int_0^{\infty} \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

3. Prove that

$$B(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n} \theta d\theta$$

Proof : By definition

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$$

Put $x = \sin^2 \theta$

$$dx = 2 \sin \theta \cos \theta d\theta$$

\therefore when $x=0$ $\theta=0$

$$x=1, \quad \begin{aligned} \sin^2 \theta &= 1 \\ \theta &= \pi/2 \end{aligned}$$

$\therefore \theta=0$ to $\theta=\pi/2$

$$\therefore B(m, n) = \int_0^{\pi/2} (\sin^2 \theta)^{m-1} (1 - \sin^2 \theta)^{n-1} 2 \sin \theta \cos \theta d\theta$$

$$= 2 \int_0^{\pi/2} \sin^{2m-2} \theta (\cos^2 \theta)^{n-1} \sin \theta \cos \theta d\theta$$

$$= 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

$$B(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n} \theta d\theta$$

$$\therefore \text{If } \Gamma(m, n) = \int_0^{\pi/2} \sin^{m-1} \theta \cos^{n-1} \theta \, d\theta,$$

$$\text{then } \Gamma(m, n) = \frac{1}{2} B\left(\frac{m}{2}, \frac{n}{2}\right)$$

4. Recurrence relation between Beta and Gamma functions:

$$\text{Prove that } \Gamma(m, n) = \frac{\Gamma(m) \Gamma(n)}{\Gamma(m+n)}$$

$$\text{Proof : } \Gamma(m) = \int_0^{\infty} e^{-t} t^{m-1} dt$$

$$\text{Put } x=t^2$$

$$dx=2t dt$$

$$\therefore \Gamma(m) = 2 \int_0^{\infty} e^{-t^2} t^{m-1} dt$$

$$= 2 \int_0^{\infty} e^{-t^2} t^{m-2+1} dt$$

$$\therefore \Gamma(m) = 2 \int_0^{\infty} e^{-x} x^{\frac{m}{2}-1} dx$$

$$\text{Similarly } \Gamma(n) = 2 \int_0^{\infty} e^{-y^2} y^{n-1} dy$$

$$\therefore \Gamma(m)\Gamma(n) = 4 \left[\int_0^{\infty} e^{-x^2} x^{m-1} dx \right] \left[\int_0^{\pi/2} \int_0^{\infty} e^{-y^2} y^{n-1} dy \, d\theta \right]$$

$$= 4 \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)} y^{n-1} x^{m-1} dx \, dy \dots\dots\dots(1)$$

Changing to polar coordinates

$$\text{Put } x=r \cos \theta \text{ and } y=r \sin \theta$$

Then $x^2+y^2=r^2$

$\theta = 0 \text{ to } \pi/2$ $r = 0 \text{ to } \infty$

$\theta = \pi/2$ $r = \infty$

$$\therefore \int_0^{\pi/2} \int_0^{\infty} (rcos\theta)^{2m-1} (rsin\theta)^{2n-1} r dr d\theta$$

$$= 4 \int_0^{\pi/2} \int_0^{\infty} r^{2m+2n-1} cos^{2m-1}\theta sin^{2n-1}\theta dr d\theta$$

$$= \left[\int_0^{\infty} r^{2m+2n-1} dr \right] \left[\int_0^{\pi/2} cos^{2m-1}\theta sin^{2n-1}\theta d\theta \right]$$

$= (m+n) B(m,n)$

$\therefore B(m,n) = \frac{(m)!(n)!}{(m+n)!}$

Note : 1. $\int_0^{\infty} e^{-r^2} n^{2n} dr = \dots$

2. $\int_0^1 (1-x)^n dx = \frac{\pi}{\sin \pi}$

22.4 Let us sum up

So far we have studied Beta, Gamma integrals and their properties.

22.5 Check your Progress

1. Find $\beta(7,8)$
2. $\Gamma(x) = \dots$
3. What is the value of $\Gamma(1/2)$
4. If n is a positive integer, $\Gamma(x+1) = \dots$

22.6 Lesson end Activities

1. Prove that $\Gamma(1/2) = \sqrt{\pi}$

2. Evaluate $\int_0^{\pi/2} \sqrt{\sin \theta} d\theta$

3. If $x > 0$, prove that $\Gamma(n+1) = n \Gamma(n)$. Deduce the value of $\Gamma(n+1)$, n is a positive integer

4. Find $\int_0^{\pi/2} \sqrt{\tan \theta} d\theta$

22.7 Points for discussion

5. Define $\beta(m, n)$

Prove $\beta(m, n) = \beta(n, m)$

6. Express $\int_0^1 x^m (1-x)^n dx$ in terms of Gamma functions and evaluate

$$\int_0^1 x^5 (1-x)^4 dx$$

22.8 References

A book of Integral Calculus by T.K. Manickavasagam Pillai

Lesson-23

CONVERGENCE OF IMPROPER INTEGRALS

Contents

- 23.0. Aims and Objectives
- 23.1 Convergence of Improper Integrals
- 23.2. Simple Examples
- 23.3. Let us sum up
- 23.4. Check your progress.
- 23.5. Lesson End Activities
- 23.6. Points for discussion
- 23.7. References

23.0. Aim and Objectives

We will learn on finding the convergence of Improper Integral

23.1 Convergence of Improper Integrals

Definition 1: The interval whose length or range is infinite is called an infinite interval.

Example : The intervals (a, ∞) , $(-\infty, b)$ and $(-\infty, \infty)$ are Infinite intervals

Definition: Proper integral. The definite integral $\int_a^b f(x) dx$ is called a proper integral if the range of integration is finite and $f(x)$ is bounded.

For example, $\int_0^{\pi/2} \sin x dx$ is a proper integral

Improper Integral : The definite integral $\int_a^b f(x) dx$ is called an improper integral if (1) the interval (a, b) is not finite or (2) the interval is finite but $f(x)$ is not bounded or (3) neither the interval (a, b) is finite nor $f(x)$ is bounded.

There are two kinds of improper integrals (viz) of first kind and second kind.

First Kind: When the range of integration is infinite and $f(x)$ is bounded.

Eg : $\int_0^{\infty} \frac{dx}{1+x^2}$ is an improper integral.

Second kind: A definite integral $\int_a^b f(x) dx$ in which the range of integration is finite but the integrand $f(x)$ is unbounded at one or more points of the interval $a \leq x \leq b$.

For example $\int_1^4 \frac{dx}{(x-2)^2}$ is an improper integral of second kind.

(ii) $\int_0^1 \frac{1}{x^2}$ is an improper integral of second kind.

Convergence of Improper integrals

The integral $\int_a^b f(x) dx$ is said to Converge to the value I, if for any arbitrarily chosen possible number ϵ , however small but not zero there exists a corresponding positive number N such that $|\int_a^b f(x) dx - I| < \epsilon$ for all values of $b \geq N$

Similarly we define the convergence of an integral, when the lower limit is infinite or when the integral becomes infinite at the upper or lower limit.

In other words, when the limit of an improper integral as defined is a definite finite number, we say that the given integral is convergent. When the limit is α or $-\alpha$, we say that the integral is called divergent if the value of the integral does not exist.

In case the limit is neither a definite number α or $-\alpha$ then the integral is called oscillatory and therefore in this case the value of the integral does not exist. the integral is not convergent.

23.2. Simple Examples

1) Evaluate $\int_1^{\infty} \frac{dx}{\sqrt{x}}$

$$I = \int_1^{\infty} \frac{dx}{\sqrt{x}}$$

$$\begin{aligned}
 &= \int_1^{\infty} x^{1/2} dx \\
 &= \lim_{x \rightarrow \infty} \int_1^x x^{1/2} dx \\
 &= \lim_{x \rightarrow \infty} \left[\frac{x^{1/2}}{1/2} \right]_1^x \\
 &= \lim_{x \rightarrow \infty} \left(\frac{2}{1} x^{1/2} - \frac{2}{1} (1)^{1/2} \right) \\
 &= \lim_{x \rightarrow \infty} (2\sqrt{x} - 2)
 \end{aligned}$$

∴ The limit does not exist finitely

The integral is divergent

(2) Evaluate $\int_3^{\infty} \frac{dx}{(x-2)^2}$

Solution $\int_3^{\infty} \frac{dx}{(x-2)^2} = \lim_{x \rightarrow \infty} \int_3^x \frac{dx}{(x-2)^2}$

$$\begin{aligned}
 &= \lim_{x \rightarrow \infty} \left[-\frac{1}{x-2} \right]_3^x \\
 &= \lim_{x \rightarrow \infty} \left(-\frac{1}{x-2} + \frac{1}{3-2} \right) \\
 &= -(-1) = 1 = 0 \text{ finite number}
 \end{aligned}$$

∴ The given integral is convergent.

(3) Test the convergence of $\int_0^{\infty} \frac{4a dx}{x^2 + 4a^2}$

Solution

$$\begin{aligned}
 \int_0^{\infty} \frac{4a dx}{x^2 + 4a^2} &= \lim_{x \rightarrow \infty} \int_0^x \frac{4a dx}{x^2 + 4a^2} \\
 &= 4a \lim_{x \rightarrow \infty} \left[\frac{1}{2a} \tan^{-1} \left(\frac{x}{2a} \right) \right]_0^x \\
 &= 2 \lim_{x \rightarrow \infty} \left[\tan^{-1} \left(\frac{x}{2a} \right) \right]_0^x \\
 &= 2 \left\{ \tan^{-1} \left(\frac{x}{2a} \right) - \tan^{-1} (0) \right\} \\
 &= 2 \frac{\pi}{2} = \pi
 \end{aligned}$$

∴ The given integral is convergent

(4) Show that $\int_0^{\infty} \frac{dx}{(1+x^2)^3}$ is not convergent

Solution

we have

$$\begin{aligned} \int_0^{\infty} \frac{dx}{(1+x^2)^3} &= \lim_{x \rightarrow \infty} \int_0^x \frac{dx}{(1+x^2)^3} \\ &= \lim_{x \rightarrow \infty} \left[\frac{x}{2(1+x^2)^2} + \frac{3x}{4(1+x^2)} + \frac{3}{8} \tan^{-1} x \right]_0^x \\ &= \lim_{x \rightarrow \infty} \left[\frac{x}{2(1+x^2)^2} + \frac{3x}{4(1+x^2)} + \frac{3}{8} \tan^{-1} x \right] \\ &= \alpha \end{aligned}$$

∴ The given integral is not convergent.

23.3. Let us sum up

We have studied on how the convergence of Improper Integrals are known.

23.4. Check your progress.

(1) Discuss the convergence of $\int_1^{\alpha} \frac{dx}{1+x^2}$

(2) Evaluate $\int_0^{\infty} e^{-x^2} dx$

23.5. Lesson End Activities

(1) Show that the Improper Integrals $\int_a^b \frac{dx}{(x-a)^n}$ converges if $n < 1$

(2) Examine the convergence of $\int_0^1 \frac{dx}{x^2 + 1}$

(3) Show that the integral $\int_{\pi}^{\infty} \frac{\sin^2 x}{x^2} dx$ is convergent.

23.6. Points for discussion

(4) Show that the integral $\int_{\pi}^{\infty} \frac{dx}{x\sqrt{1+x^2}}$

(5) Test the convergence of $\int_{\pi}^{\infty} \frac{\sin^2 x}{x^2} dx$

(6) Test the convergence of $\int_{\pi}^{\infty} \frac{dx}{\sqrt{x^3+1}}$

(7) Test the convergence of $\int_2^{\infty} \frac{dx}{2\sqrt{x^2-1}}$

23.7. References

A book of Integral Calculus by Vasiswaha and Sharma

Lesson – 24

APPROXIMATE INTEGRATION

Contents

- 24.0. Aims and Objectives
- 24.1. Simpson’s rule:
- 24.2. Examples
- 24.3. Let us sum up
- 24.4. Check your progress
- 24.5. Lesson End Activities
- 24.6. Points for Discussion
- 24.7. References

24.0 Aims and Objectives

We shall study to find out the area of curve $y=f(x)$ between $x=a$ and $x=b$ by using (1) Simpson’s rule (2) Trapezoidal rule

Approximate Integration

We know that the area between the curve $y=f(x)$, the x axis and the ordinates $x=a, x=b$ is given by $\int_a^b f(x) dx$. This method will fail when the equating of the curve is not available or the integral is not integrable.

The simplest method of approximating to the area would be: (a) Simpson’s rule (b) Trapezoidal rule.

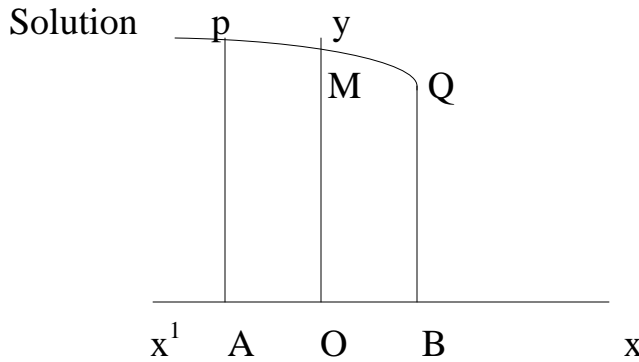
24.1. Simpson’s rule:

To evaluate $\int_a^b f(x) dx$

Divide the interval $[a,b]$ into n equal parts let the length of each sub-interval be h . Let $y_1, y_2, y_3, \dots, y_{2n+1}$ be the values of the function to be integrated at $2n+1$ equidistant points applying the parabolic rule to the white area

$$\int_a^b f(x) dx \approx \frac{1}{3} h [y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5 + \dots + 4y_{2n} + y_{2n+1}]$$

(a) Simpson's one-third rule or parabolic rule



We shall first find the area ABQP let $AO=OB=h$

Through any three points, a parabola can pass through, having its axis parallel to of axis.

Let the centum of such parabola passing through P,M and Q be $y=a+bx+cx^2$

Taking the are ABQP to be approximately given by the area enclosed by this

parabola, the x axis and the ordinates AP, BQ, aera ABQP = $\int_{-h}^h y d$

$$\begin{aligned}
 &= \left[a + \frac{b^2 x}{2} + \frac{c^3 x^3}{3} \right]_{-h}^h \\
 &= \left(a + \frac{b^2 h}{2} + \frac{c^3 h^3}{3} \right) - \left(a - \frac{b^2 h}{2} + \frac{c^3 h^3}{3} \right) \\
 &= a + \frac{b^2 h}{2} + \frac{c^3 h^3}{3} + a - \frac{b^2 h}{2} + \frac{c^3 h^3}{3} \\
 &= 2 \left(a + \frac{c^3 h^3}{3} \right) \\
 &= \frac{2}{3} h [3a + ch^2]
 \end{aligned}$$

Let y_1, y_2, y_3 , be the lengths of the ordinates at A,O,B respectively

$$y_1 = a-bh+ch^2; y_2=a; y_3 = a+bh+ch^2$$

$$\therefore a = y_2$$

$$\therefore y_1 = y_2-bh+ch^2$$

$$y_3 = y_2+bh+ch^2$$

$$\therefore y_1+y_3 = 2(y_2+ch^2)$$

$$\frac{y_1 + y_2}{2} = y^2 \cdot c^2 h$$

$$\therefore c_2 = \frac{y_1 + y_2}{2} \cdot y$$

$$\therefore \text{Area ABQP} = \frac{2}{3} h (c_1 + c_2)$$

$$= \frac{2}{3} h \left[\frac{1}{2} (y_1 + y_2) + y_2 \right]$$

$$= \frac{h}{3} [6y_2 + y_1 + y_2]$$

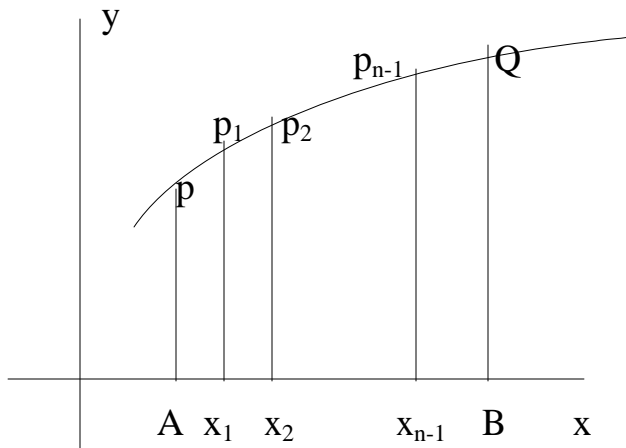
$$= \frac{h}{3} [y_1 + 4y_2 + y_3]$$

This approximation is called one third rule or parabolic rule

(c) Trapezoidal rule

To evaluate $\int_a^b y \, dx$

Divide the integral $[a,b]$ into n equal parts at $x_1, x_2, x_3, \dots, x_{n-1}$



Erect the ordinates at $A, x_1, x_2, x_3, \dots, x_{n-1}, B$.

Join the chords $pp_1, p_1p_2, p_2p_3, \dots, p_{n-1}Q$

Take the sum of the areas of the trapezium $Ax_1p_1p, x_1x_2, p_2p_1, \dots, x_{n-1}BQp_{n-1}$ as the value of the area.

Area of the trape $Ax_1p_1p = \frac{1}{2}h(y_1 + y_2)$ etc

$$\begin{aligned} \therefore \text{Area of } ABQP &= \int_a^b y \, dx \\ &= \frac{1}{2}h[(y_1 + y_2) + (y_2 + y_3) + \dots + (y_{n-1} + y_n)] \\ &= \frac{1}{2}h[y_1 + 2y_2 + 2y_3 + \dots + 2y_{n-1} + y_n] \end{aligned}$$

24.2. Examples

Evaluate $\int_1^4 \frac{dx}{x}$ taking 6 equal intervals. Hence find \log_2 to 5 decimal places. (a) by Trapezoidal rule (b) by simpson's rule

Solution

Step 1

$$f(x) = 1/x$$

$$a=1, b=4$$

$$h = \frac{b-a}{n} = \frac{4-1}{6} = \frac{3}{6} = \frac{1}{2}$$

x	1	1.5	2	2.5	3	3.5	4
---	---	-----	---	-----	---	-----	---

1/x	1	0.66666	0.5	0.4	0.33333	0.28571	0.25
-----	---	---------	-----	-----	---------	---------	------

$$y_1=1, y_2=0.66666; y_3=0.5, y_4=0.4, y_5=0.33333; y_6=0.28571; y_7=0.25$$

(a) By Trapezoidal rule

$$\int_1^4 \frac{1}{x} dx = \frac{1}{2} [(y_1 + y_7)h + 2(y_2 + y_4)h + 4(y_3 + y_5)h]$$

$$= \frac{1}{2} \times 0.25 [(1 + 0.25) + 2(0.66666 + 0.4) + 4(0.5 + 0.33333)]$$

$$\log_4 = 1.358925$$

$$\log_2^2 = 1.358925$$

$$\therefore 2 \log_2 = 1.358925$$

$$\therefore \log_2 = 0.6794625$$

(b) Using Simpson's rule

$$\int_1^4 \frac{1}{x} dx = \frac{1}{3} [(y_1 + y_7)h + 4(y_2 + y_6)h + 2(y_3 + y_5)h + 4y_4h]$$

$$= \frac{1}{6} [(1 + 0.25) + 4(0.66666 + 0.28571) + 2(0.5 + 0.33333) + 4 \times 0.4]$$

$$= \frac{1}{6} [(1.25) + 4(0.95237) + 2(0.83333) + 1.6]$$

$$\log_4 = 1.38769$$

$$\log_2^2 = 1.38769$$

$$2 \log_2 = 1.38769$$

$$\log_2 = 0.693845$$

24.3. Let us sum up

We have seen how to find area of a curve $y=f(x)$ between specified limits using Simpson's rule and Trapezoidal rule.

24.4. Check your progress

(1) Evaluate $\int_0^\pi \sin x dx$ by Simpson's rule for $x=0, \frac{\pi}{1}, \frac{2\pi}{1}, \dots, \pi$

(Nov 2000)

(2) Evaluate $\int_0^1 \frac{1}{1+x} dx$ using Trapezoidal rule with $h=0.125$ (Apr 2003)

(3) Compute $\int_0^{\infty} \frac{x \, d}{1+x^2}$ by Simpson's rule taking eight intervals

(Bharathiar Ap 2006, Ap 2005)

24.5. Lesson End Activities

(1) Evaluate $\int_1^4 \frac{1}{x^2} \, d$, Taking 4 equal intervals, using simpson's rule.

(Ans : 16.70)

(2) Evaluate $\int_1^1 \frac{1}{x^3} \, d$, taking 10 equal intervals using trapezoidal rule

(Ans : 3690)

24.6. Points for Discussion

(1) Applying Simpson's rule to find $\int_0^1 \frac{d}{1+x}$, correct to three places of decimals, taking $h=0.25$. (Ap 2003)

(2) Using Trapezoidal rule, find $\int_0^{\pi} \sin d$, taking 10 equal parts. (Ap 2000).

(3) Complete $\int_0^{\infty} \frac{x \, d}{1+x^2}$ by Simpson's rule taking eight intervals

(Bharathiar Ap 2006, Ap 2005)

24.7 References

A book of Integral Calculus by Vasiswaha and Sharma

Lesson – 25

CHANGE OF ORDER OF INTEGRATION

Contents

- 25.0 Aims and Objectives
- 25.1. Change of order of integration in double Integral
- 25.2. Examples – 1.
- 25.3. Let us sum up
- 25.4. Check your progress
- 25.5. Lesson End Activities
- 25.6. Points for discussion
- 25.7. Improper Integral
- 25.8. References

25.0 Aims and Objectives

In this lesson we shall study the evaluation of double integral using change of order of integration.

25.1. Change of order of integration in double Integral

When the limits for inner integration are functions of a variable, the change in order of integration will result in changes in the limit of integration i.e. the double integral $\int_a^b \int_g^h f(x, y) dx dy$ will take the form

$$\int_c^d \int_{f_1(x)}^{f_2(y)} g(x, y) dy dx$$

When the order of integration is changed.

This process of converting a given double integral into an equivalent double integral is called change of order or integration.

Working Rule : (1) The region of integration is identified first

- (2) a rough sketch of the region is drawn
- (3) New limits are fixed
- (4) Then integral is evaluated.

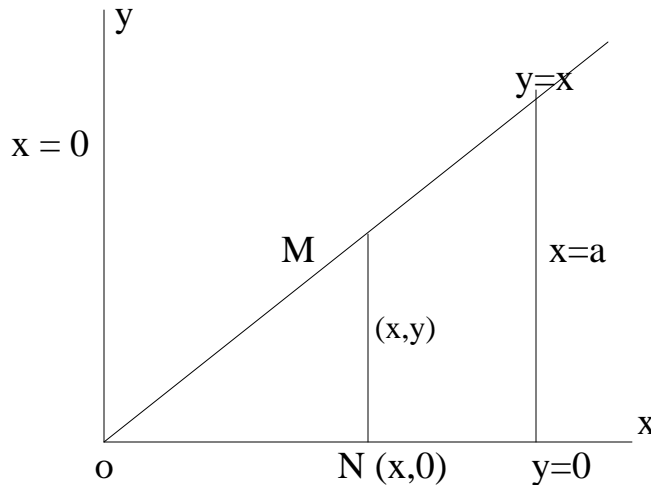
25.2. Examples – 1.

Change the order of integration $\int_0^a \int_y^a \frac{x}{\sqrt{x^2 + y^2}} dx dy$ and then evaluate

the same.

Solution : The region of integration is

$$y \leq x \leq a \text{ and } 0 \leq y \leq a$$



Let R denote the region bounded by the lines $x=y$, $x=a$, $y=0$, $y=a$.

$$\therefore I = \int_R \int \frac{y^x}{\sqrt{x^2 + y^2}} dx$$

By treating x as a constant, i.e. we shall draw a line parallel to y axis in the region R of integration.

$$\begin{aligned} \therefore I &= \int_0^a \int_{y=0}^{y=x} \frac{y^x}{\sqrt{x^2 + y^2}} dy dx \\ &= \int_0^a x \left[1 + y \circ \sqrt{x^2 + y^2} \right]_{y=0}^{y=x} dx \\ &= \int_0^a x \left[1 + x \circ \sqrt{x^2 + (x)^2} \right] dx \\ &= \int_0^a x \left[1 + \frac{(x + 2\sqrt{2}x)}{2} \right] dx \\ &= \int_0^a x \left[1 + \sqrt{2}x \right] dx \\ &= \left[\frac{x^2}{2} + \frac{\sqrt{2}x^3}{3} \right]_0^a \\ &= \left(\frac{a^2}{2} + \frac{\sqrt{2}a^3}{3} \right) \end{aligned}$$

$$= \frac{a^2}{2} \log(\sqrt{1-z})$$

2. Change the order of integration in $\int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy dx dy$ and then evaluate the

same

Solution :

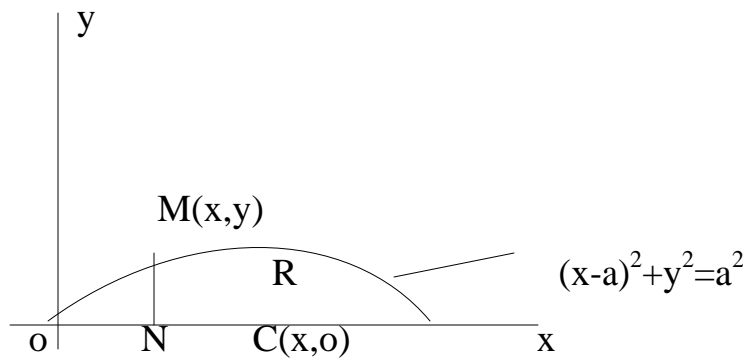
We shall fix the region of integration R as $x = a \pm \sqrt{a^2 - y^2}$

i.e. $x - a = \pm \sqrt{a^2 - y^2}$

$$(x-a)^2 = a^2 - y^2$$

$$\therefore (x-a)^2 + (y^2) = a^2$$

Which is a circle with the centre at (a,0) and radius a unit.



Keeping x as a Constant, draw a line MN to y axis $M(x, \sqrt{a^2 - x^2})$; $x=0$ to $2a$
 $y=0$ to $y= \sqrt{a^2 - x^2}$

After changing the order of integration

$$\begin{aligned} I &= \int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} xy dx dy \\ &= \int_{x=0}^{2a} \int_{y=0}^{\sqrt{x^2-x^2}} xy dy dx \\ &= \int_{x=0}^{2a} x \left(\frac{y^2}{2} \right)_0^{\sqrt{x^2-x^2}} dx \\ &= \frac{1}{2} \int_0^{2a} x(x^2 - x^2) dx \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \int_0^{2a} (ax^2 + x^3) dx \\
 &= \frac{1}{2} \left[\frac{2ax^3}{3} + \frac{x^4}{4} \right]_0^{2a} \\
 &= \frac{1}{2} \left[\frac{2a(8a^3)}{3} + \frac{(2a)^4}{4} \right] \\
 &= \frac{1}{2} a^4 \left(\frac{16}{3} + 4 \right) \\
 &= \frac{2}{3} a^4
 \end{aligned}$$

25.3. Let us sum up

We have studied so far the evaluation of double integral using changing order of integration.

25.4. Check your progress

(1) Change the order of integration in the integral $\int_0^a \int_{x^2/a}^x xy \, dx \, dy$ and evaluate it. (Ap 2003) (Ap 2005)

(2) Change the order of integration and evaluate $\int_0^a \int_x^a (x^2 + y^2) \, dy \, dx$ (Ap 2003)

(3) By changing the order of integration evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx$
Ap 2005

25.5. Lesson End Activities

Change of order of integrals and convergence of Improper Integrals.

(1) Change the order of integration

$$\int_0^1 \int_x^1 \frac{dx \, dy}{x^2 + y^2} \text{ and then evaluate the same}$$

(Ans : $\frac{1}{2} \log 2$)

(2) Change the order of integration

$$\int_0^b \int_0^{a/y} xy \, dx \, dy \text{ and evaluate } \left(\frac{a^2}{2} \log \frac{b}{a} \right)$$

(3) Change the order of integration and evaluate.

$$\int_0^2 \int_0^2 (x^2 + y) \, dy \, dx \quad (\text{Ans } 16/3)$$

25.6. Points for discussion

(4) Change of the order of integration and evaluate

$$\int_0^1 \int_{y^2}^y \frac{y \, dx \, dy}{x^2 + y} \quad (\text{Ans : } \frac{1}{2} \log 2)$$

Exercise : Convergence of

25.7. Improper Integral

(1) Show that $\int_1^{\infty} \frac{dx}{x^{3/2}}$ is convergent

(2) Show that $\int_1^{\infty} \frac{dx}{x^{3/2}}$ is divergent

(3) Show that $\int_0^{\infty} e^{-mx} \, dx$ is convergent

(4) Test the Convergence of $\int_0^{\infty} e^{2k} \, dx$ (divergent)

(5) Test convergence of the following

(a) $\int_{-\alpha}^{\infty} e^x \, dx$ (b) $\int_{-\alpha}^0 e^{-x} \, dx$

(c) $\int_{-\alpha}^0 \sin x \, dx$ (d) $\int_{-\alpha}^0 \cos x \, dx$

(e) $\int_{-\alpha}^{\infty} e^{-x} \, dx$ (f) $\int_{-\alpha}^{\infty} \frac{dx}{1+x^2}$

(g) $\int_{-\alpha}^{\infty} \frac{dx}{x^2 + 2x + 2}$

- Answer (a) Convergent (b) Divergent
 (c) Divergent (d) divergent
 (e) divergent (f) convergent
 (g) convergent

25.8. References

A book of Integral Calculus by Vasiswaha and Sharma

Lesson – 26

JACOBIANS

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- 26.0. Aims and Objectives
- 26.1. JACOBIANS
- 26.2. Three variables
- 26.3. Change of variable in case of Two variables
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- 26.5. Examples Two variables
- 26.6. Let us sum up
- 26.7 Lesson End Activities
- 26.8. Points for discussion
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26.0. Aims and Objectives

Our aim is to determine Jacobians using partial derivatives and the concept of a determinant.

26.1. JACOBIANS

Two Variables

If $u=f(x,y)$, $v=g(x,y)$ be two continuous functions of the independent

variables x and y , then $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ is called the Jacobian of u and v w.r.t x and y

and is denoted by $J\left(\frac{u, v}{x, y}\right)$

26.2. Three variables

Let u, v, w be functions of x, y, z the Jacobian of u, v and w w.r.t. x, y, z is defined by

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$\begin{vmatrix} \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

and is denoted by $J\left(\frac{u, v, w}{x, y, z}\right)$

$$\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \frac{\partial w}{\partial z} \quad \text{or} = \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

Note : Jacobians will be useful while evaluating multiple integrals which is an easy way of evaluation.

Result 1: Let u,v, be function of x and y, whose x and y functions of α,β, then

$$\frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(\alpha, \beta)} = \frac{\partial(u, v)}{\partial(\alpha, \beta)} \dots\dots(1)$$

Result 2 Prove that $\frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(u, v)} = 1$

Proof : In the above result (1) Put α =u; β=x we have.

$$\frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial(u, v)}{\partial(u, v)}$$

But $\frac{\partial(u, v)}{\partial(u, v)} = \frac{\partial u}{\partial x} \frac{\partial v}{\partial u}$

But u and v are independent variables,

$$\frac{\partial u}{\partial v} = 0 \quad \frac{\partial v}{\partial x} = 0$$

using in (2)

$$\therefore \frac{\partial(u, v)}{\partial(u, v)} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\therefore \frac{\partial(u, v)}{\partial(x, y)} \frac{\partial(x, y)}{\partial(u, v)} = 1$$

26.3. Change of variable in case of Two variables

Formout1. Suppose we want to charge the variable in the integral

$$\int_R f(x, y) dx dy \text{ to } u, v \text{ by the relation } x=f(u, v); y=\phi(u, v)$$

we have

$$d = x \frac{\partial x}{\partial y} d + u \frac{\partial x}{\partial y} d \dots ($$

$$d = y \frac{\partial y}{\partial x} d + u \frac{\partial y}{\partial v} d \dots ($$

We shall first treat y, a constant and integrate w.r.t.x.

$$dy=0$$

$$\therefore (2) \text{ becomes } 0 = \frac{\partial y}{\partial x} d - \frac{\partial y}{\partial v} d$$

$$\therefore \frac{\partial y}{\partial x} = \frac{\partial y}{\partial v} \dots v$$

$$- \frac{\partial y}{\partial u} d = \frac{\partial y}{\partial v} d$$

\therefore using in (1)

$$d = x \frac{\partial x}{\partial u} d - x \frac{\partial u}{\partial v} \frac{\left(\frac{\partial y}{\partial u} d \right)_x}{\frac{\partial y}{\partial u}}$$

$$= \frac{\frac{\partial x}{\partial u} \frac{\partial y}{\partial v} \frac{\partial y}{\partial v} \frac{\partial u}{\partial v} \cdot x}{\frac{\partial y}{\partial v} / \frac{\partial y}{\partial v}}$$

$$= \frac{\frac{\partial(x, y)}{\partial(v, u)} d}{\frac{\partial y}{\partial x}}$$

$$\therefore \int_R f(x) \frac{\partial(x, y)}{\partial(v, u)} d \int d y \frac{\partial(u, v)}{\partial y} d$$

Changing the order of integration

$$= \int d \int_1 (f, y) \frac{\partial(x, y)}{\partial y} \frac{\partial(u, v)}{\partial v} d \cdot y \quad (3)$$

Eliminate v, we get a relation between u,x and y \therefore x can be expressed in terms of x and y.

$$\therefore f(x,y) \text{ is transformed into } f_1(x,y)$$

To integrate (3), u is considered as a constant $dx=0$

$$\therefore \text{ from (2) } d = \frac{\partial y}{\partial v} dv$$

$$\begin{aligned} \therefore (3) \text{ becomes } \int d \frac{f_1(u, v, \frac{\partial(x,y)}{\partial(u,v)})}{\frac{\partial y}{\partial v}} &= \int f_2(u, v, \frac{\partial(x,y)}{\partial(u,v)}) dv \\ &= \int f_2(u, v, \frac{\partial(x,y)}{\partial(u,v)}) dv \end{aligned}$$

26.4. Three variables

In case of three variables u,v,w

Which are functions of x,y,z the Jacobian of u,v,w, w.r.t x,y,z is defined by

$$J \begin{pmatrix} u, v, w \\ x, y, z \end{pmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$J \begin{pmatrix} u, v, w \\ x, y, z \end{pmatrix} \text{ is also denoted by } \frac{\partial(u, v, w)}{\partial(x, y, z)}$$

26.5. Examples Two variables

$$J \begin{pmatrix} u, v \\ x, y \end{pmatrix} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$$

1) If $x+y=u$, $x-y=v$, then show that

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{2}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$u = x+y$$

$$v = x-y$$

Add $u+v=2x$; $x=1/2 (u+v)$

subtract, $u-v=2y$ $y=1/2 (u-v)$

$$\frac{\partial x}{\partial u} = 1/2 \qquad \frac{\partial y}{\partial v} = 1/2$$

$$\frac{\partial x}{\partial v} = 1/2 \qquad \frac{\partial y}{\partial u} = -1/2$$

$$J \left(\begin{matrix} x, y \\ u, v \end{matrix} \right) = \frac{1}{2} \quad \frac{1}{2}$$

$$\frac{1}{2} \quad \frac{1}{2}$$

$$= -1/4 \quad 1/4 = -1/2$$

(2) If $x+y+z=u$; $y+z=uv$; $z=uvw$,

Show that $\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2$. (Nov 2006 Bharathiar)

Solution

$$u=x+y+z \dots\dots(1)$$

$$uv=y+z \dots\dots(2)$$

$$uvw = z \dots\dots(3)$$

$$uv=y+uvw$$

$$y=uv-uvw$$

using (2) in (1)

$$u=x+uv$$

$$x=u-uv$$

$x = u - uv$ $\frac{\partial x}{\partial u} = 1 - v$ $\frac{\partial x}{\partial v} = -u$ $\frac{\partial x}{\partial w} = 0$	$y = uv - uvw$ $\frac{\partial y}{\partial u} = v - vw$ $\frac{\partial y}{\partial v} = u - uw$ $\frac{\partial y}{\partial w} = -uv$	$z = uvw$ $\frac{\partial z}{\partial u} = vw$ $\frac{\partial z}{\partial v} = uw$ $\frac{\partial z}{\partial w} = uv$
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1-v	v-vw	vw
u	u-uw	uw
0	-uv	uv

Expanding through first column

$$J = (1-v) \begin{vmatrix} u-uw & uw \\ -uv & uv \end{vmatrix} + u \begin{vmatrix} v-vw & vw \\ -uv & uv \end{vmatrix}$$

$$= (1-v)(uv) \begin{vmatrix} u-uw & uw \\ -1 & 1 \end{vmatrix} + u \cdot uv \begin{vmatrix} v-vw & vw \\ -1 & 1 \end{vmatrix}$$

$$= (1-v)u(v-uw) + u^2(v-v/w + w/v - w^2/v)$$

$$= (1-v)u^2v + u^2v^2$$

$$= u^2v^2 + u^2v$$

$$= u^2v$$

26.6. Let us sum up

so far we have studied the method of finding Jacobians for two variables and three variables.

Check your progress

(1) If $x=uv^2$, $y=u^2v$, Show that

$$\frac{\partial(x, y)}{\partial(u, v)} = 3u^2v$$

(2) If $x=r \cos v$, $y=r \sin v$,

show that $\frac{\partial(x, y)}{\partial(u, v)} = r$

26.7 Lesson End Activities

(1) If $x^2 - y^2 = u$, $xy = v$ show that

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{x^2 - y^2}$$

(2) If $x = v \sin v \cos \phi$

$$y = v \sin v \sin \phi$$

$$z = r \cos v, \text{ prove that } J = r^2 \sin v$$

(3) If $x+y=u$, $y=uv$ find $\frac{\partial(x, y)}{\partial(u, v)}$ (Ans :u)

(4) If $u=x^2-y^2$; $v=x^2+y^2$, prove that

$$\frac{\partial(x, y)}{\partial(u, v)}$$

26.8. Points for discussion

(5) If $\frac{\partial(x, y)}{\partial(u, v)} = u$, then find the value of

$$\frac{\partial(x, y)}{\partial(u, v)}$$

(6) If $u = \frac{y}{x}v^z = \frac{z}{y}w^x = \frac{x}{z}w^y$

find $\frac{\partial(u, v, w)}{\partial(x, y, z)}$ (Apr. 2006)

(7) If $u = x + y$, $v = x - y$, then find $\frac{\partial(u, v)}{\partial(x, y)}$

26.9. References

A book of Integral Calculus by T.K. Manickavasagam Pillai